Berkeley Math Circle: Monthly Contest 6

Due March 1, 2016

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 6 is due on March 1, 2016.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 6, Problem 3
  Bart Simpson
  Grade 5, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules. Enjoy solving these problems and good luck!

Problems for Contest 6

1. A fair coin is flipped nine times. Which is more likely, having exactly four heads or having exactly five heads?

2. Let a and b be positive real numbers. Prove that

$$\sqrt{a^2 - ab + b^2} \geq \frac{a+b}{2}.$$

3. Let A and B be two points on the plane with $AB = 7$. What is the set of points P such that $PA^2 = PB^2 - 7$?

4. The numbers 1, 2, …, 50 are written on a blackboard. We may erase two numbers a and b, and replace both with $a + b + 2ab$; we repeat this operation until only one number remains. Prove that the value of this last number does not depend on how the operations were performed.

5. Show that $\sin 10^\circ$ is irrational.

6. Let $c > 0$ be a positive real number. We define the sequence (x_n) by $x_0 = 0$ and

$$x_{n+1} = x_n^2 + c$$

for each $n \geq 0$. For which values of c is it true that $|x_n| < 2016$ for all n?

7. Let ABC be a triangle, and let X, Y, Z be the excenters opposite A, B, C. The incircle of triangle ABC touches BC, CA, AB at points D, E, F. Finally, let I and O denote the incenter and circumcenter of triangle ABC.

   Prove that lines DX, EY, FZ, IO are concurrent.