Berkeley Math Circle: Monthly Contest 5 Solutions

1. Let A, B, C be points in a line on this order, with $AB = BC$. A car travels from A to B at 40 kilometers per hour and then from B to C at 60 kilometers per hour, without stopping in between. What is the overall speed of the trip from A to C?

   Solution. Let $d = AB = BC$. Then the total time taking is $\frac{d}{40} + \frac{d}{60}$, and the total distance is $2d$. Hence the overall speed is

   $$\frac{2d}{\frac{d}{40} + \frac{d}{60}} = \frac{2}{\frac{1}{40} + \frac{1}{60}} = 48.$$

   □

2. For positive integers n, prove that $\gcd(6n + 1, 15n + 2) = 1$.

   Solution. Let $a = 6n + 1$ and $b = 15n + 2$, and notice that $5a - 2b = 5(6n + 1) - 2(15n + 2) = 1$. Thus, if a positive integer d divides both a and b, we must have d dividing $5a - 2b = 1$ as well. This implies $d = 1$, as desired. □

3. A line in the Cartesian plane is called *stable* if it passes through at least two points (x, y) such that x and y are rational numbers. Prove or disprove: every point lies on some stable line.

   Solution. The assertion is false: we will show that the point $(\sqrt{2}, \sqrt{3})$ does not lie on a stable line.

   Note that the slope of any stable line must be a rational number. Now assume for contradiction that (a, b) lies on a stable line through $(\sqrt{2}, \sqrt{3})$, where a and b are both rational. Then $\frac{\sqrt{2}-a}{\sqrt{3}-b} = m$ for some rational number m, which leads us to

   $$mb - a = m\sqrt{3} - \sqrt{2}.$$

   Since $\sqrt{2}$ is not rational, we must have $m = 0$. Then, squaring both sides gives

   $$(mb - a)^2 = (3m^2 + 2) - 2m\sqrt{6}.$$

   Since $m \neq 0$ this implies $\sqrt{6}$ is irrational, which is a contradiction. □

4. There are three boxes of stones. Each hour, Sisyphus moves a stones from one box to another. For each transfer of a stone, he receives from Zeus a number of coins equal to the number of stones in the box from which the stone is drawn minus the number of stones in the recipient box, with the stone Sisyphus just carried not counted. If this number is negative, Sisyphus pays the corresponding amount (and can pay later if he is broke).

   After 1000 years, all the stones lie in their initial boxes. What is the greatest possible earning of Sisyphus at that moment?

Solution. The myth of Sisyphus suggests the answer: 0.

Let x be the amount of money Sisyphus has and a, b, c the sizes of the boxes. The key observation is that the quantity

$$N = 2x + a^2 + b^2 + c^2$$

does not change; for example, after one operation from the first box to the second the quantity becomes

$$N' = 2(x+a-b-1)+(a-1)^2+(b+1)^2+c^2 = 2x+2a-2b-2+a^2-2a+1+b^2+2b+1+c^2 = 2x+a^2+b^2+c^2 =$$

Thus the quantity N is an *invariant.*

Consequently, if Sisyphus has no money to begin with then he does not earn any money at the end, either. □

5. Let a, b, c be positive reals. Prove that

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c.$$

Solution. By the AM-GM inequality, we have

$$\frac{2\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab}}{4} \geq \sqrt[4]{\frac{a^3 \cdot a^3 \cdot b^3 \cdot c^3}{bc \cdot bc \cdot ca \cdot ab}} = a.$$

Adding the analogous two inequalities yields the conclusion. □

6. Let $\mathcal{P}$ be a regular 17-gon; we draw in the $\binom{17}{2}$ diagonals and sides of $\mathcal{P}$ and paint each side or diagonal one of eight different colors. Suppose that there is no triangle (with vertices among vertices of $\mathcal{P}$) whose three edges all have the same color. What is the maximum possible number of triangles, all of whose edges have *different* colors?

Solution. We approach the problem by finding the minimum number of triangles with a pair of edges of the same color; we call such triangles "isosceles". We can count the number of such triangles by instead considering:

> The number of isosceles triangles is equal to the number of pairs of adjacent edges of the same color.

So, if we let $n_{i,v}$ denote the number of edges of the ith color touching vertex v, the number of isosceles triangles can be written as

$$\sum_{v \text{ vertex}} \sum_{i=1}^{8} \binom{n_{i,v}}{2}.$$

Since $\sum_{i=1}^{8} n_{i,v} = 16$, by Jensen's Inequality we actually have

$$\sum_{v \text{ vertex}} \sum_{i=1}^{8} \binom{n_{i,v}}{2} \geq \sum_{v \text{ vertex}} 8\binom{16/8}{2} = 17 \cdot 8 = 136.$$

So, the number of triangles with all different colors is at most $\binom{17}{3} - 17 \cdot 8 = 544$. We leave the construction of a maximal example as an exercise to the reader. □

7. Let ABC be an acute triangle with orthocenter H and altitudes BD, CE. The circumcircle of ADE cuts the circumcircle of ABC at $F \neq A$. Prove that the angle bisectors of $\angle BFC$ and $\angle BHC$ concur at a point on BC.

Solution. Since F is the Miquel point of the complete quadrilateral $BDEC$, it follows that $\triangle FDB \sim \triangle FEC$ (indeed one can see this by angle chasing). Note also that triangles BDH and CEH are similar. Consequently,

$$\frac{FB}{FC} = \frac{DB}{EC} = \frac{HB}{HC}$$

and thus the angle bisector theorem completes the solution. □