Berkeley Math Circle: Monthly Contest 4

Due January 12, 2016

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 4 is due on January 12, 2016.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 4, Problem 3
  Bart Simpson
  Grade 5, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules. Enjoy solving these problems and good luck!

Problems for Contest 4

1. A thief starts at the point 0 on a number line, and runs either left or right at a constant rate. Two hours later, a policeman which moves twice as fast as the robber appears. However, the policeman does not know which direction the robber went and cannot see the robber. How can the policeman catch the robber?

2. Determine, with proof, the value of $\log_2 3 \log_3 4 \log_4 5 \ldots \log_{255} 256$.

3. Let $ABCD$ be a convex quadrilateral. For which point P in the plane is the sum $PA + PB + PC + PD$ minimized?

4. Let a, b, c be positive integers such that $a^2 - bc$ is a perfect square. Prove that the number $2a + b + c$ is not a prime number.

5. For positive integers $x_1, x_2, \ldots, x_n$ satisfying $x_1 + \cdots + x_n = 101n$, prove that

$$\binom{x_1}{2} + \binom{x_2}{2} + \cdots + \binom{x_n}{2} \geq 5050n.$$

6. In triangle ABC, a point M is selected in its interior so that $\angle MAB = 10^\circ$, $\angle MBA = 20^\circ$, $\angle MCA = 30^\circ$ and $\angle MAC = 40^\circ$. Determine the value of $\angle MBC$.

7. A thief starts at the point 0 on a number line, and runs either left or right at a constant rate. Many hours later, a policeman which moves twice as fast as the robber appears. However, the policeman does not know which direction the robber went and cannot see the robber; moreover, the policeman does not know how long ago the robber departed. How can the policeman catch the robber?