Berkeley Math Circle: Monthly Contest 3 Solutions

1. Compute the prime factorization of $25^3 - 27^2$.

   Solution. Noticing that $25 = 5^2$ and $27 = 3^3$, we have this is

$$\begin{aligned} 5^6 - 3^6 &= (5^3 - 3^3)(5^3 + 3^3) \\ &= (5 - 3)(5^2 + 5 \cdot 3 + 3^2)(5 + 3)(5^2 - 5 \cdot 3 + 3^2) \\ &= 2 \cdot 49 \cdot 8 \cdot 19 \\ &= 2^4 \cdot 7^2 \cdot 19. \end{aligned}$$

   □

2. Suppose x and y are real numbers satisfying $x + y = 5$. What is the largest possible value of $x^2 + 2xy$?

   Solution. The quantity in question is $(x+y)^2 - y^2 \le (x+y)^2 = 25$. Equality occurs when $x = 5$ and $y = 0$, hence the maximum possible value is 25. □

3. Let ABC be an acute triangle with orthocenter H, circumcenter O, and incenter I. Prove that ray AI bisects $\angle HAO$.

   Solution. Without loss of generality, $AB < AC$. It follows that $\angle BAH = 90^\circ - \angle B$, since the extension of AH is perpendicular to BC. Moreover, we also have $\angle AOC = 2\angle B$; but since $OA = OC$, this implies $\angle OAC = \frac{1}{2}(180^\circ - \angle AOC) = 90^\circ - \angle B$. So we conclude that $\angle BAH = \angle CAO$. Since $\angle BAI = \angle CAI$ as well, it follows that $\angle HAI = \angle OAI$, which is what we wanted to prove. □

4. For which prime numbers p is $p^2 + 2$ also prime? Prove your answer.

   Solution. The answer is $p = 3$. This indeed works, since $3^2 + 2 = 11$.

   Consider any other prime number $p \ne 3$. Then it follows that $p^2 \equiv 1 \pmod 3$; i.e. that p leaves remainder 1 when divided by 3. Consequently, $p^2 + 2$ is divisible by 3. Since $p \ge 2$, we have $p^2 + 2 \ge 7$ as well, thus $p^2 + 2$ cannot be prime in this case. □

5. There is a colony consisting of 100 cells. Every minute, a cell dies with probability $\frac{1}{3}$; otherwise it splits into two identical copies. What is the probability that the colony never goes extinct?

   Solution. The answer is $1 - (\frac{1}{2})^{100}$.

   Let p be the probability that a colony consisting of just one cell will survive. Then

$$p = \frac{1}{3} \cdot 0 + \frac{2}{3}\left(1 - (1-p)^2\right)$$

   owing to the fact that when the cell splits in two, the probability both of them go extinct is $(1-p)^2$. Solving for p, we obtain $p = \frac{1}{2}$.

The colony initially has 100 cells; if we treat these as 100 distinct colonies, we obtain the claimed answer. ☐

6. Let H, I, O, Ω denote the orthocenter, incenter, circumcenter and circumcircle of a scalene acute triangle ABC. Prove that if $\angle BAC = 60^\circ$ then the circumcenter of $\triangle IHO$ lies on Ω.

   Solution. First, we show that the five points B, O, H, I, C all lie on a circle. To see this, note that

$$\begin{aligned} \angle BIC &= 90^\circ + \frac{1}{2}\angle BAC = 120^\circ \\ \angle BOC &= 2\angle BAC = 120^\circ \\ \angle BHC &= 180^\circ - \angle BAC = 120^\circ. \end{aligned}$$

   So, this proves the claim.

   Let M be the midpoint of arc BC of Ω now (not containing A). Evidently, $MB = MC$ and $\angle BMC = 120^\circ$. Since $OB = OC$ and $\angle BOC = 120^\circ$ as well, we discover that triangles BMO and CMO are actually equilateral triangles, whence $MB = MO = MC$; i.e. M is the circumcenter of $\triangle BOC$. Since B, O, H, I, C are all concyclic, M is the circumcenter of $\triangle IHO$ as well, as desired. ☐

7. Let a, b, c be positive integers. Prove that it is not possible for a^2+b+c, b^2+c+a, c^2+a+b to all be perfect squares.

   Solution. Without loss of generality we may assume $\max\{a, b, c\} = a$. Then

$$a^2 < a^2 + b + c \le a^2 + 2a < (a+1)^2.$$

   So, a^2+b+c is not a perfect square, because it lies strictly between two perfect squares. ☐

8. Let n be a fixed positive integer. Initially, n 1's are written on a blackboard. Every minute, David picks two numbers x and y written on the blackboard, erases them, and writes the number $(x+y)^4$ on the blackboard. Show that after $n-1$ minutes, the number written on the blackboard is at least $2^{\frac{4n^2-4}{3}}$.

   Solution. We proceed by strong induction n, with the base case $n = 1$ being vacuous. For the inductive step, consider the situation in which we have two numbers x and y remaining on the blackboard. Suppose the first one was written after $a-1$ operations, and the second one was written after $b-1$ operations, so that $(a-1)+(b-1) = n-2$. Then by the inductive hypothesis,

$$x \ge 2^{\frac{4a^2-4}{3}}, \qquad y \ge 2^{\frac{4b^2-4}{3}}.$$

   Consequently, using convexity and the bound $(a+b)^2 \le 2(a^2+b^2)$, we have

$$x + y \ge 2 \cdot 2^{\frac{2(a^2+b^2)-4}{3}} \ge 2^{\frac{(a+b)^2-1}{3}} = 2^{\frac{n^2-1}{3}}.$$

   So $(x+y)^4 \ge 2^{\frac{4n^2-4}{3}}$ as needed. ☐