Berkeley Math Circle: Monthly Contest 2 Solutions

1. Let $s_1, s_2, \ldots$ be an infinite arithmetic progression of distinct positive integers. Prove that $s_{s_1}, s_{s_2}, \ldots$ is also an infinite arithmetic progression of distinct positive integers.

   Solution. Let $s_n = an + b$ for some integers a and b. Then $s_{s_n} = a(an+b)+b = a^2n + (ab+b)$, which is also an arithmetic progression. □

2. Is there a polynomial $P(n)$ with integer coefficients such that $P(2) = 4$ and $P(P(2)) = 7$? Prove your answer.

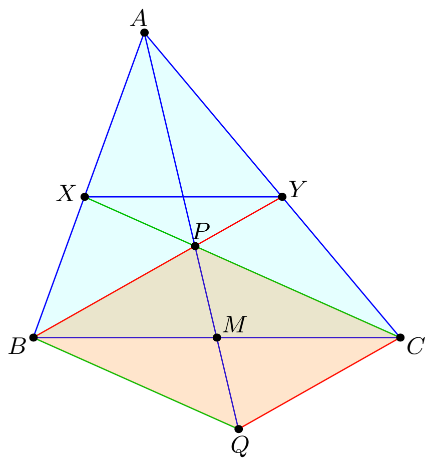
   Solution. The answer is no. Let $P(n) = c_n x^n + \cdots + c_0$. We are given that $P(2) = 4$ and $P(4) = 7$. The first equation implies that c_0 is even while the second implies that c_0 is odd, which is a contradiction. □

3. Are there integers a, b, c, d which satisfy $a^4 + b^4 + c^4 + 2016 = 10d$?

   Solution. The answer is no. Look at the equation in base 5. Observe that $0^4 = 0$, $1^4 = 1 = 1_5$, $2^4 = 16 = 31_5$, $3^4 = 81 = 311_5$, $4^4 = 256 = 2011_5$, so each of a^4, b^4, c^4 must end in 0 or 1 in base 5. On the other hand $10d - 2016$ ends with 4 in base 5. This is impossible. □

4. Let ABC be a triangle and P a point inside it. Rays BP and CP meet AC and AB at Y and X, respectively. Prove that if AP bisects BC then $XY \parallel BC$.

   Solution. Let Q be the reflection of P across M (with M the midpoint of BC). Accordingly, $BPCQ$ is a parallelogram.



   From this, we see that $\triangle AXP \sim \triangle ABQ$ and $\triangle AYP \sim \triangle ACQ$, and thus we deduce

$$\frac{AX}{AB} = \frac{AP}{AQ} = \frac{AY}{AC}$$

   so $XY \parallel BC$. □

5. Yan and Jacob play the following game. Yan shows Jacob a weighted 4-sided die labelled 1, 2, 3, 4, with weights $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{7}$, $\frac{1}{42}$, respectively. Then, Jacob specifies 4 positive real numbers x_1, x_2, x_3, x_4 such that $x_1 + \cdots + x_4 = 1$. Finally, Yan rolls the dice, and Jacob earns $10 + \log(x_k)$ dollars if the die shows k (note this may be negative). Which x_i should Jacob pick to maximize his expected payoff?

   (Here log is the natural logarithm, which has base $e \approx 2.718$.)

   Solution. Jacob should pick $(x_1, x_2, x_3, x_4) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{7}, \frac{1}{42})$. More generally, suppose the weights are $p_1, \ldots, p_4$. Then Jacob's expected payoff is

$$10 + \sum_{i=1}^{4} p_i \log(x_i) = 10 + \sum_{i=1}^{4} p_i \log p_i + \sum_{k=1}^{4} p_i \log\left(\frac{x_i}{p_i}\right).$$

   Now, by Jensen's Inequality on the concave function $\log x$, we obtain

$$\sum_{i=1}^{4} p_i \log\left(\frac{x_i}{p_i}\right). \leq \log\left(\sum_{i=1}^{4} p_i \cdot \frac{x_i}{p_i}\right) = \log 1 = 0$$

   and equality occurs exactly when $\frac{x_1}{p_1} = \frac{x_2}{p_2} = \frac{x_3}{p_3} = \frac{x_4}{p_4}$; that is, when $x_i = p_i$ for every i. □

6. Let $X = \{1, 2, \ldots, 100\}$. How many functions $f : X \to X$ satisfy $f(b) < f(a) + (b-a)$ for all $1 \leq a < b \leq 100$?

   Solution. The answer is $\binom{199}{100}$. We claim that the functions are precisely those of the form $f(n) = n + a_n$, where

$$-99 \leq a_{100} < a_{99} < \cdots < a_1 \leq 99$$

   is an arbitrary sequence. The answer follows from this.

   To see that all functions are of this form, we rewrite the given as $f(b) - b < f(a) - a$, which tells us that $f(100) - 100 < f(99) - 99 < \cdots < f(1) - 1$. Since $f(100) - 100 \geq -99$ and $f(1) - 1 \leq 99$, this shows all functions are of the form claimed above, i.e. that $1 - n \leq f(n) - n \leq 100 - n$.

   Finally, it remains to check that all functions of the form satisfy the conditions. The inequality $f(b) < f(a) + (b - a)$ is immediate. Moreover, it is easy to see that $a_{100} \geq -99$, $a_{99} \geq -98$, and so on, so $1 \leq n + a_n$ holds; similarly, $n + a_n \leq 100$ holds too. Thus $n + a_n$ is indeed an element of X. □

7. Find, with proof, the largest possible value of

$$\frac{x_1^2 + \cdots + x_n^2}{n}$$

   where real numbers $x_1, \ldots, x_n \geq -1$ are satisfying $x_1^3 + \cdots + x_n^3 = 0$.

   Solution. For any i, we have $0 \leq (x_i + 1)(x_i - 2)^2 = x_i^3 - 3x_i^2 + 4$. Adding all of these we deduce that $\sum_{i=1}^{n} x_i^2 \leq \frac{1}{3} \sum_{i=1}^{n} (x_i^3 + 4) = \frac{4}{3} n$. Equality occurs, for example, when $n = 9$, $x_1 = \cdots = x_8 = -1$ and $x_9 = 2$. Therefore, the answer is $\frac{4}{3}$. □