Berkeley Math Circle: Monthly Contest 2

Due November 3, 2015

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 2 is due on November 3, 2015.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 2, Problem 3
  Bart Simpson
  Grade 5, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules. Enjoy solving these problems and good luck!

Problems for Contest 2

1. Let $s_1, s_2, \ldots$ be an infinite arithmetic progression of distinct positive integers. Prove that $s_{s_1}, s_{s_2}, \ldots$ is also an infinite arithmetic progression of distinct positive integers.

2. Is there a polynomial $P(n)$ with integer coefficients such that $P(2) = 4$ and $P(P(2)) = 7$? Prove your answer.

3. Are there integers a, b, c, d which satisfy $a^4 + b^4 + c^4 + 2016 = 10d$?

4. Let ABC be a triangle and P a point inside it. Rays BP and CP meet AC and AB at Y and X, respectively. Prove that if AP bisects BC then $XY \parallel BC$.

5. Yan and Jacob play the following game. Yan shows Jacob a weighted 4-sided die labelled 1, 2, 3, 4, with weights $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{7}$, $\frac{1}{42}$, respectively. Then, Jacob specifies 4 positive real numbers x_1, x_2, x_3, x_4 such that $x_1 + \cdots + x_4 = 1$. Finally, Yan rolls the dice, and Jacob earns $10 + \log(x_k)$ dollars if the die shows k (note this may be negative). Which x_i should Jacob pick to maximize his expected payoff?

   (Here log is the natural logarithm, which has base $e \approx 2.718$.)

6. Let $X = \{1, 2, \ldots, 100\}$. How many functions $f : X \to X$ satisfy $f(b) < f(a) + (b - a)$ for all $1 \le a < b \le 100$?

7. Find, with proof, the largest possible value of

$$\frac{x_1^2 + \cdots + x_n^2}{n}$$

where real numbers $x_1, \ldots, x_n \ge -1$ are satisfying $x_1^3 + \cdots + x_n^3 = 0$.