BMC Intermediate II: Triangle Centers

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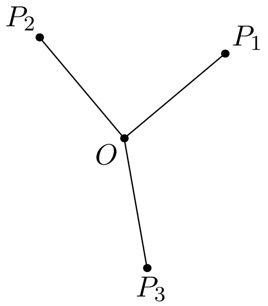
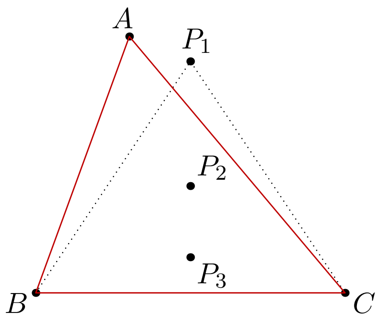
1 The Eyeballing Game

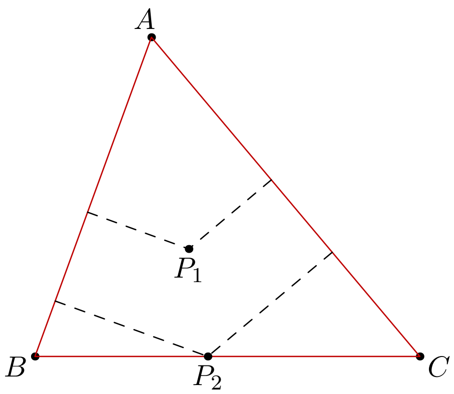
Game 0. Given a point O, the locus of points P such that the length of OP is 5.



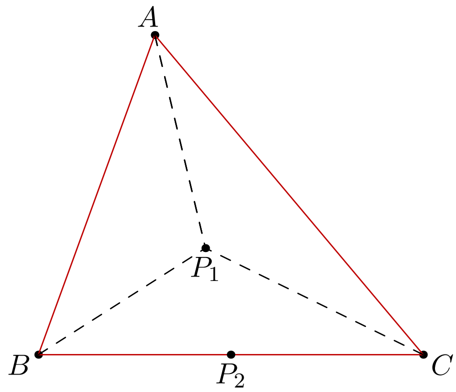
Game 1. Given a triangle ABC, the locus of points P such that $PB = PC$.



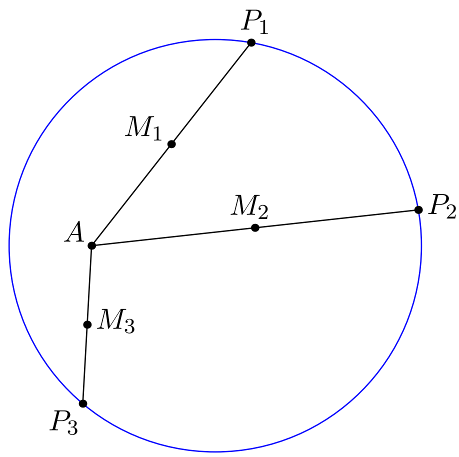
Game 2. Given a triangle ABC, the locus of points P in $\triangle ABC$ such that the distance from P to $\overline{AB}$ and $\overline{AC}$ is the same.



Game 3. Given a triangle ABC, the locus of points P such that the triangles PAB and PAC have the same area. (This one's a little harder, you might want to skip it and come back later.)

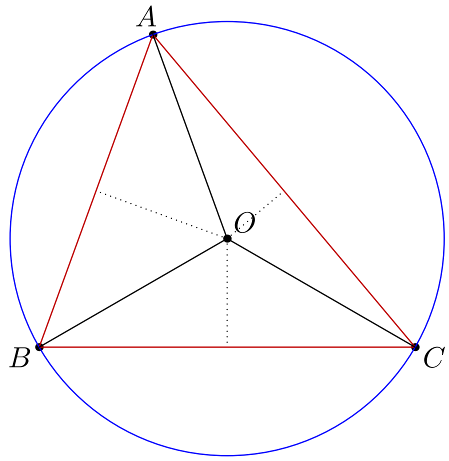


Game 4. Let A be a point and consider a circle. Let P be any point on the circle, and M the midpoint of $\overline{AP}$. As P varies, what is the locus of M?

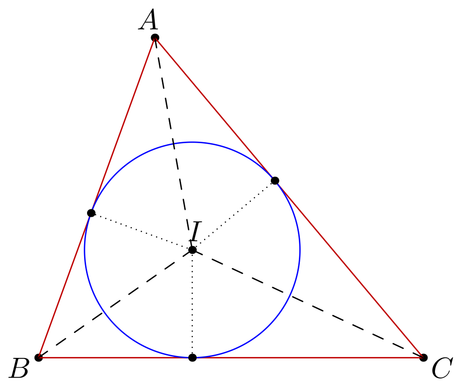


2 Triangle Centers

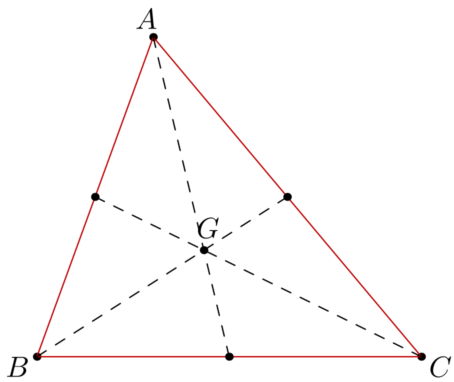
Circumcenter O (via Game 1)



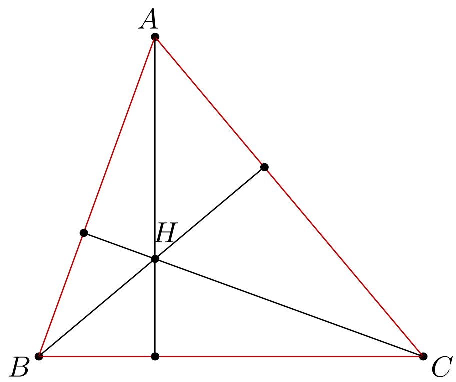
Incenter I (via Game 2)



Centroid G (via Game 3)



Orthocenter H (via Game 1! How?)

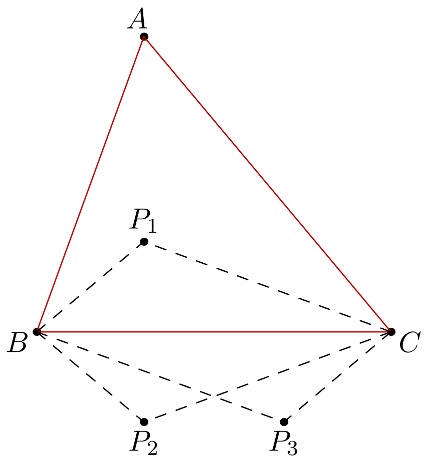


In words:

- Every triangle has exactly one circle passing through all of its vertices. This is called the **circumcircle**, and its center O is called the **circumcenter**. The point O is the intersection of the perpendicular bisectors.
- Every triangle has exactly one circle which is inscribed inside it. This is called the **incircle**, and the center I is called the **incenter**. It is the intersection of the angle bisectors.
- The three **medians** of a triangle concur at a center G called the **centroid** of the triangle.
- The three **altitudes** of a triangle concur at a center H called the **orthocenter** of the triangle.

3 Orthocenter Flipping

Game 5. Given a triangle ABC, the points P such that $\angle BPC = 180° - \angle A$.



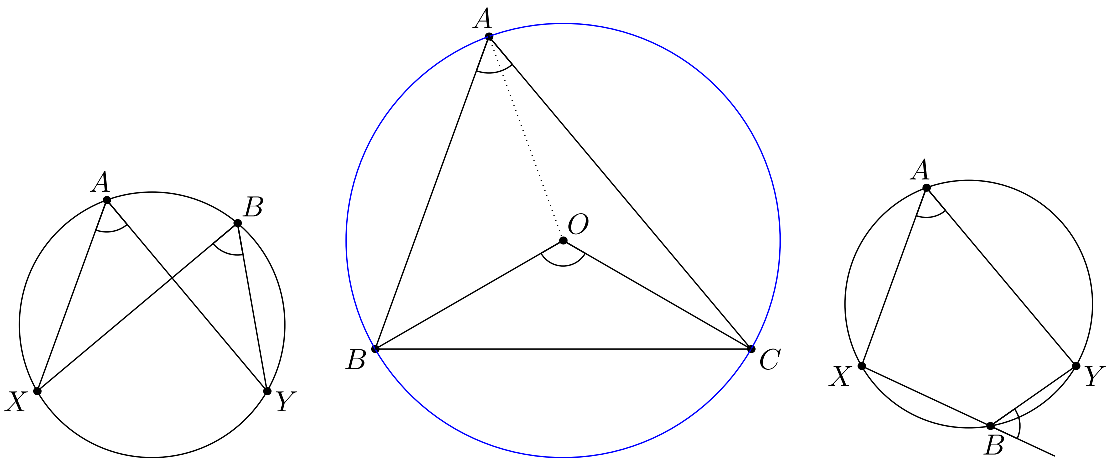
Some hints for the above game:

- Can you identify the point P_1 I've drawn?
- How are P_2 and P_3 related to P_1?
- What's the relation between A, B, P_2, P_3, and C?

4 Circle Theorems

Theorem 1. *The sum of the angles in a triangle is* $180°$.

Theorem 2 (Inscribed Angle Theorem, also "Star Trek Theorem")**.** *An inscribed angle in a circle cuts out an arc equal to twice its measure.*



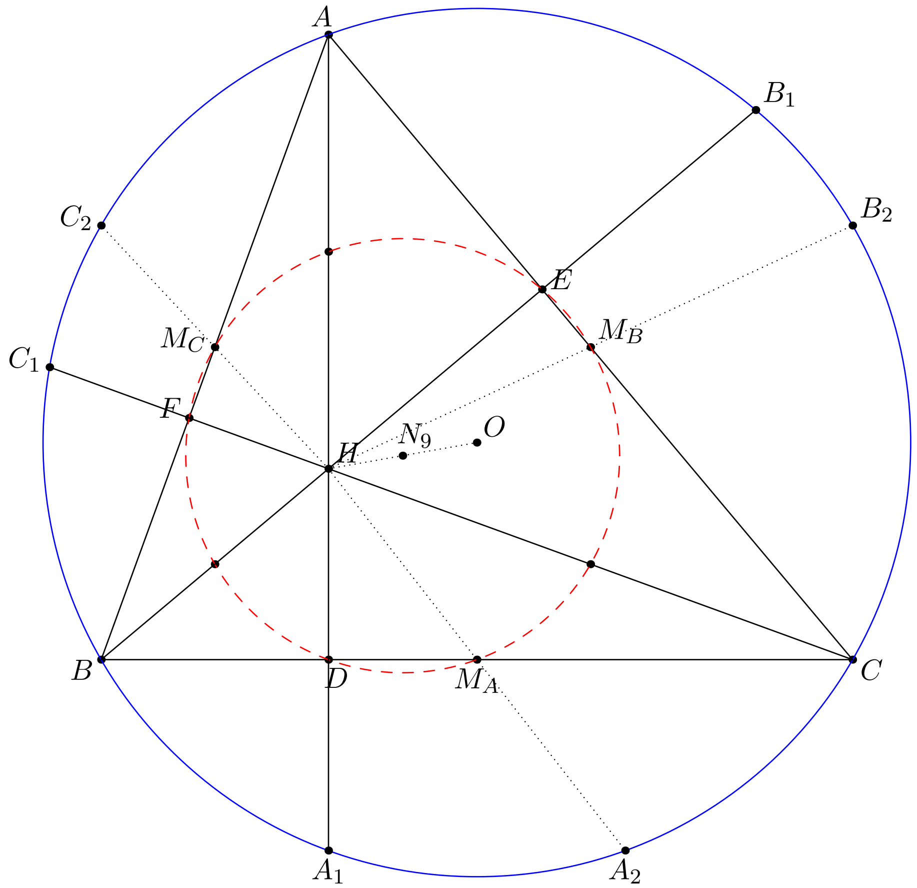
The inscribed angle theorem has important special cases. As in the left diagram, suppose A, B, Y, X lie on a circle in that order. Then $\angle XAY = \angle XBY$, because both are equal to $\frac{1}{2}\widehat{XY}$! Similarly, if A, X, B, Y lie on a circle in that order, then $\angle XAY + \angle XBY = 180°$, because the two angles mark out the entire circle: the sum of those arcs is $360°$.

It turns out the reverse is true: if points A, B, X, Y satisfy the conditions above, all four lie on a circle.

5 The Nine-Point Circle

What happens when we combine Game 4 and Game 5?

Theorem 3 (Nine-Point Circle)**.** *Let ABC be a triangle with orthocenter H. The feet of the altitudes, the midpoints of the sides, as well as the three midpoints of $\overline{AH}$, $\overline{BH}$, $\overline{CH}$ all lie on a single circle, called the* ***nine-point circle*** *of triangle ABC. Its center is the midpoint of $\overline{OH}$.*

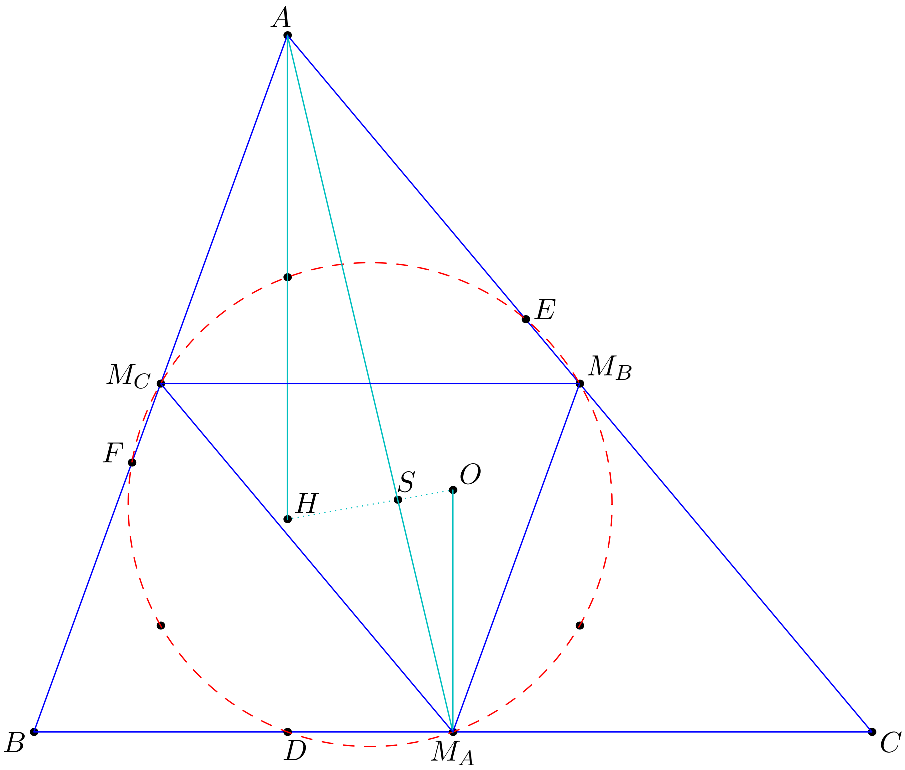


Summary of work:

- $\angle BHC = 180^\circ - \angle A$.
- As we saw in Game 5, the reflections of H across $\overline{BC}$ and its midpoint all lie on the circumcircle of $\triangle ABC$.
- Now apply Game 4.

6 Euler's Line

Now let $\overline{AM_A}$ intersect $\overline{OH}$ at S.



- Why are triangles AHS and OM_AS similar?
- Why are triangles ABC and $M_AM_BM_C$ similar? What ratio?
- Which center of $M_AM_BM_C$ is O? (Not the circumcenter!)
- Compute AH/OM_A.
- Compute HS/SO and AS/SM_A.

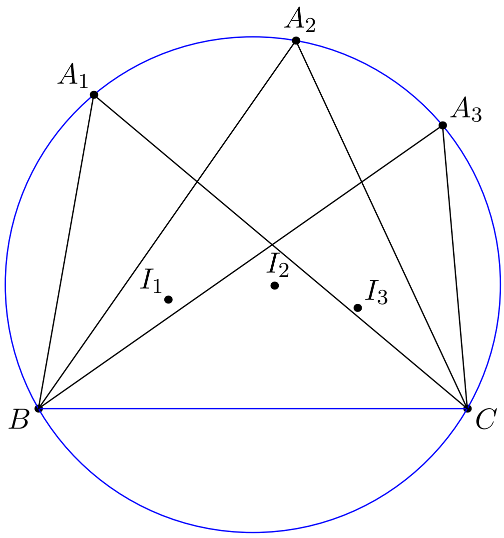
So let's give S a new definition: it is the "$\frac{1}{3}$-way point of O and H". Our work above tells us that A, S, M_A are collinear. That means BM_B and CM_C must also pass through it.

So, what is S?

Theorem 4 (Euler Line)**.** *The centroid, circumcenter, orthocenter, and nine-point center are all collinear.*

7 The Incenter

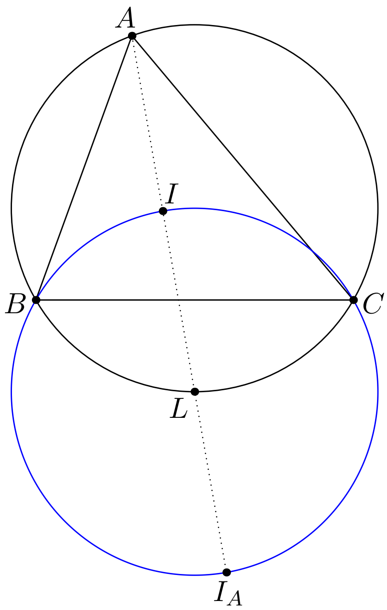
Game 6. Let ABC be a triangle with circumcircle Γ and incenter I. We hold Γ, B, C fixed and let A vary on Γ. What is the locus of I?



Can you prove it?

To fully appreciate the result of this problem, I define the A-excenter I_A to be the intersection of the *external angles* of $\angle B$ and $\angle C$. Then we get the following result.

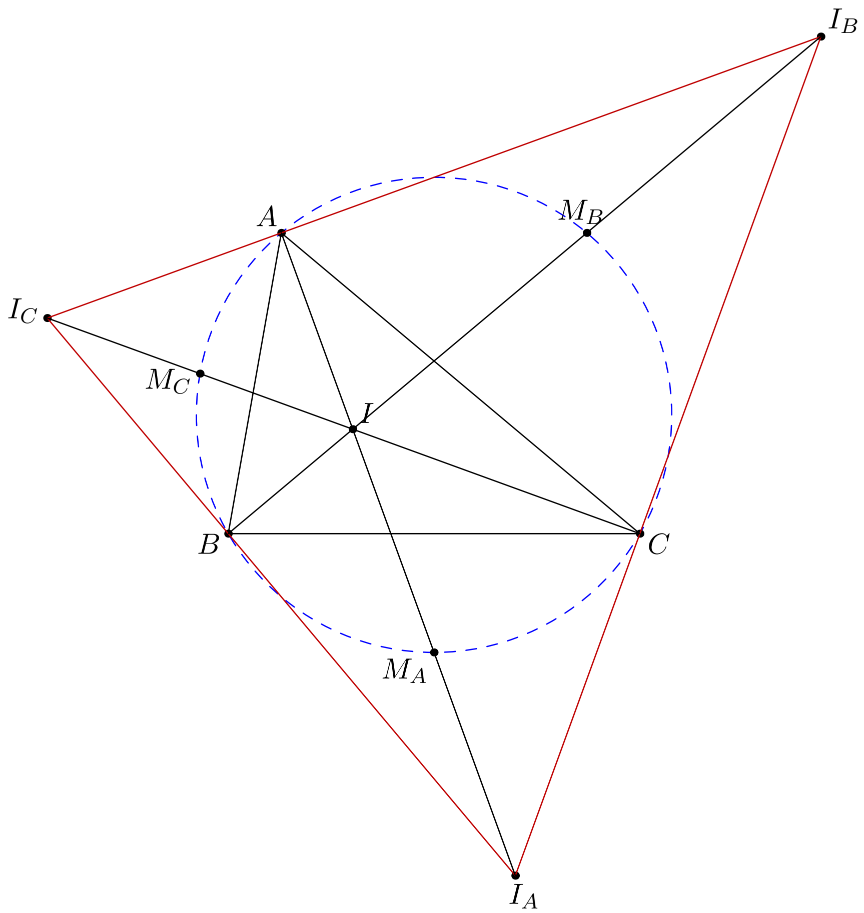
Theorem 5 (Incenter/Excenter Lemma)**.** *Let ABC be a triangle with incenter I, A-excenter I_A, and denote by L the midpoint of arc BC. Then L is the center of a circle through I, I_A, B, C.*



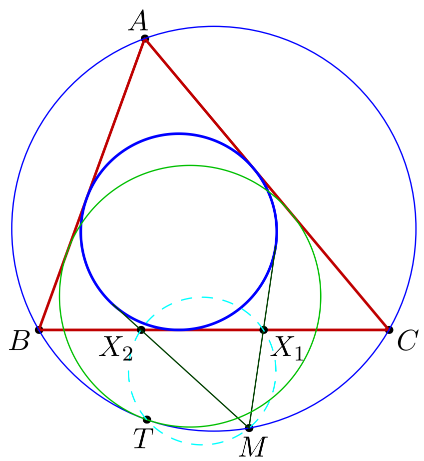
You can do this entirely by just computing angles. Try it!

8 Tying It All Together

What happens if we add in all the excenters?

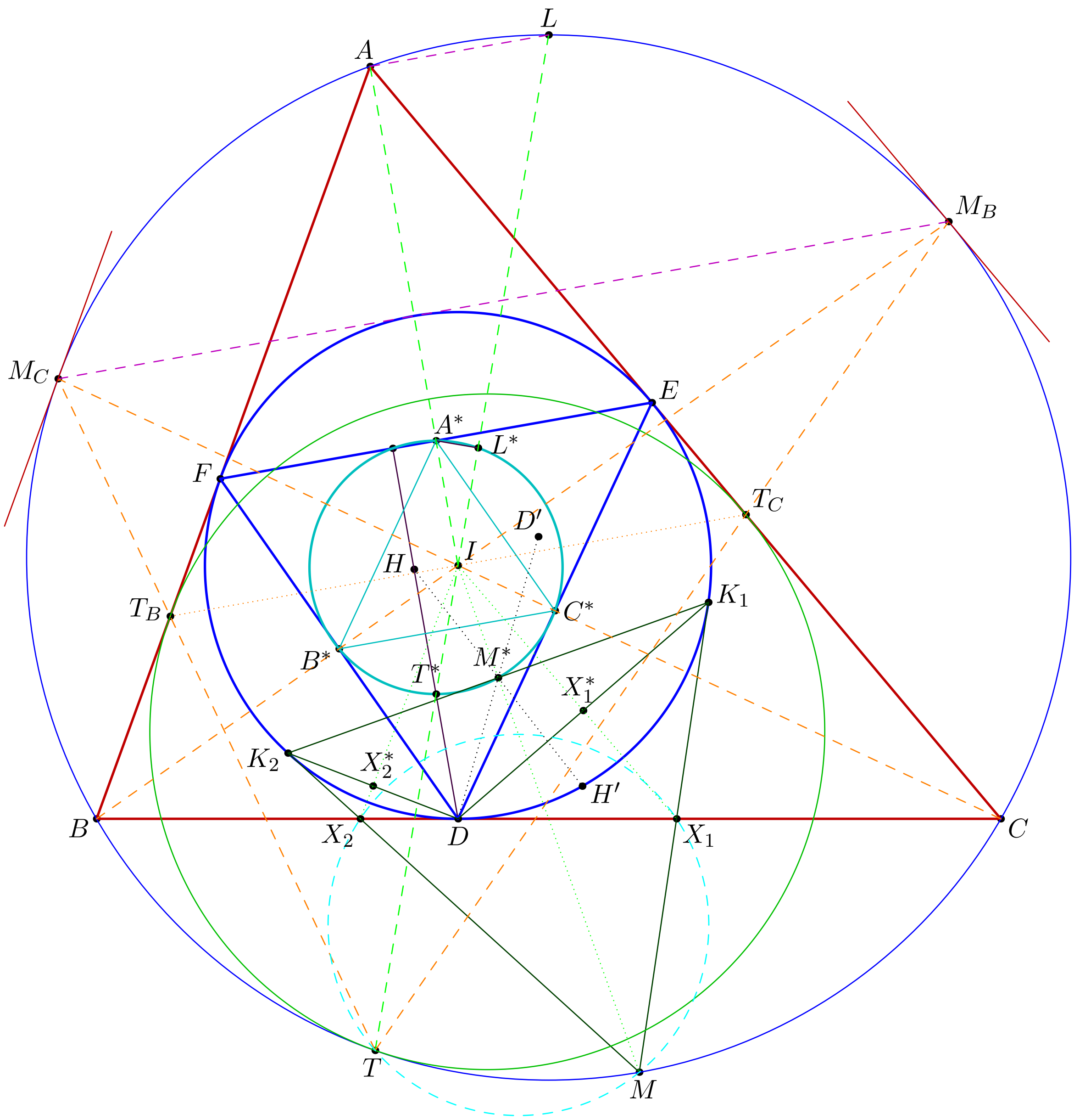


9 A Problem To Think About If You're Bored



Problem. Let ABC be a triangle and let M be a point on its circumcircle. The tangents from M to the incircle of ABC intersect at points X_1 and X_2. A circle ω (called the A-mixtilinear incircle) touches $\overline{AB}$ and $\overline{AC}$, as well as the circumcircle internally at the point T. Prove that points T, M, X_1, X_2 lie on a circle.

Here is the diagram for the solution I gave. See if you can find some of the things we talked about hiding in this picture! (For those of you that were here in October, there's also Pascal's Theorem hiding somewhere in there.)



This problem is by Cosmin Poahata. It appeared as the most difficult problem of the 2014 Taiwan Team Selection Tests for the International Mathematical Olympiad. No one solved it during the time limit, although we discovered the above solution afterwards.