BMC INTERMEDIATE II: KNOT THEORY

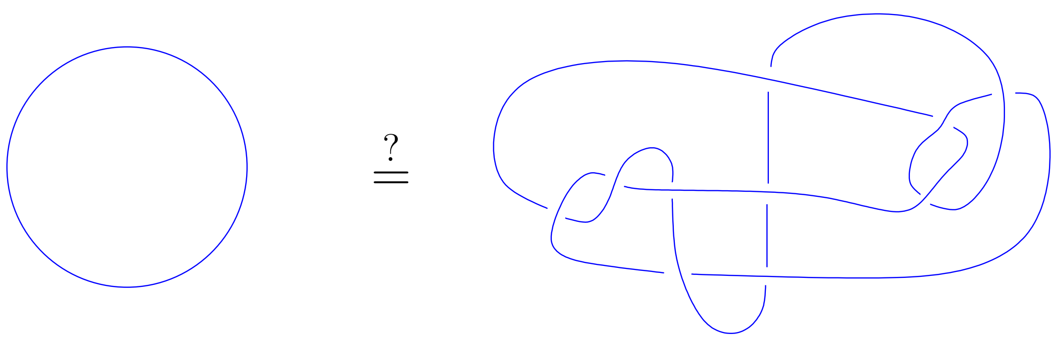
ZSUZSANNA DANCSO

1. Knots

In mathematics, more precisely topology, a *knot* is a knotted string whose ends have been glued together. (Why is just a knotted string not interesting to topologists? If the string is open, the knot can always be undone.) Two knots are "the same", if you can pull, stretch, deform one (but without cutting the string), to match the other. In topology this is called an *isotopy*, and instead of saying that two knots are the same, we say they are *isotopic* or have the same *isotopy type*. Two knots that are isotopic can *look* very different though.

How do you draw knots in 2D? The string will sometimes cross over itself, and we draw the lower string broken to indicate this. This is called a knot diagram. The most basic question of knot theory is how to decide whether two knot diagrams represent the same knot. Even if the diagrams look very different, this may or may not be the case.

Question 1. Do the following two knot diagrams represent the same knot?



"unknot" something crazy

When exactly do two knot diagrams represent isotopic knots? What are the simplest changes (or *moves*) that you can perform on a knot diagram that do not change the isotopy type?

The *Reidemeister Theorem* tells us that there are three such basic moves (called Reidemeister moves), and two knot diagrams represent isotopic knots if and only if you can make one match the other by a sequence of Reidemeister moves. (We'll discuss what the Reidemeister moves are in class.)

Question 2. In Figure 1 (on the next page), each knot in the top row is isotopic to one in the bottom row. Match them up, and try to keep track of each Reidemeister move you use.

Even though we know that two knot diagrams represent isotopic knots precisely if they can be made the same via Reidemeister moves, this doesn't necessarily make it easy to decide whether two knots are isotopic. What Reidemeister moves do you use exactly? In what order? This can be a very hard question if the knots are big and complicated. Even worse, how would you ever prove that two knots are *not* isotopic? For example, can we be absolutely

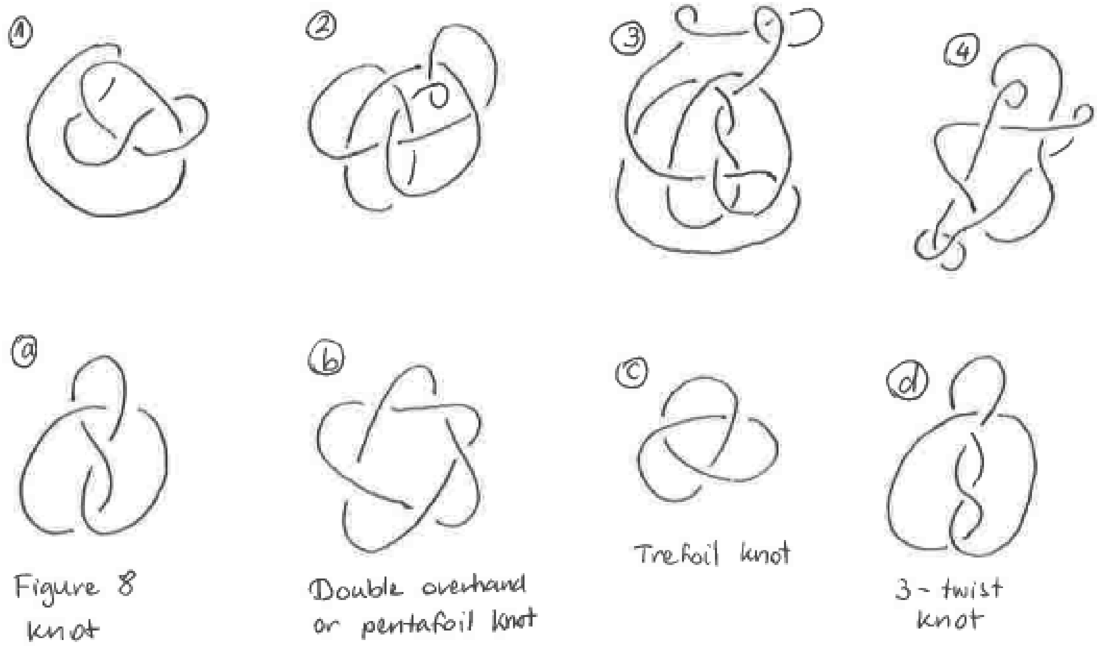


FIGURE 1

sure that the double overhand knot and the 3-twist knot above are different? Maybe we are just not smart enough to find the Reidemeister moves that match them...

2. Knot invariants

A *knot invariant* assigns a number to every knot (or knot diagram), so that if you perform an isotopy (or Reidemeister moves), then that number stays the same. More generally, the value of the invariant may not be a number but instead a polynomial, or a True/False value, or something else; it is an invariant as long as isotopies do not change the value.

A simple knot-invariant we'll study now is called *three-coloring*. Try to color each arc of a knot diagram with one of three colors (red, green and blue), so that at each crossing the three arcs involved are either all the same or all different colors. If this condition is satisfied, then we call the coloring a good 3-coloring of the knot diagram.

Question 3. What is the simplest good 3-coloring for any knot diagram? (This is called a *trivial* coloring.)

A knot diagram is called 3-colorable if it has a *non-trivial* good 3-coloring.

Question 4. Which of the knot diagrams a-d in the second row of Question 2 are 3-colorable? Is the unknot 3-colorable?

Theorem. *3-colorability is an isotopy invariant. I.e., if a knot diagram is 3-colorable and you do a Reidemeister move, the new diagram will still be 3-colorable.*

Question 5. Prove this theorem.

Question 6. Prove that the trefoil knot (c) in Figure 1 is not isotopic to the unknot, or any of the knots (a), (b) or (d).

Question 7. How many good 3-colorings are there for the trefoil knot diagram?

Question 8. Prove that for any given knot diagram, the number of good 3-colorings is always divisible by 3.
Question 9. Prove that the *number of* good 3-colorings is also an isotopy invariant.
Question 10. Prove that if the "colors" used for 3-coloring are 0, 1 and 2, then the modulo 3 sum of two good 3-colorings is also a good 3-coloring. Also, prove that if you multiply a good 3-coloring by 0, 1 or 2 modulo 3 you'll always get another good 3-coloring.
Question 11. (Big challenge) Prove that the number of good three colorings of any knot diagram is always a power of 3.
Question 12. (Challenge) Try to find two 3-colourable knots that you can still distinguish (prove they are not isotopic) because they have a different *number* of good 3-colourings.