

Regular and Semi-Regular Polyhedra
Berkeley Math Circle
September 2, 2014

1. What are the angles of a regular n -gon?
2. What are the angles of a regular star n -gon?
3. What is the area of a star polygon?

Platonic solids

4. Classify regular polyhedra.
5. For each regular polyhedron, calculate the number of vertices, edges and faces.
6. Prove that $V - E + F = 2$ for any convex polyhedron. This is *Euler's formula*.

Semi-regular polyhedra

7. Show there exist semi-regular polyhedra with vertex configuration 3.4.3.4 and 3.5.3.5. These are called the cuboctahedron and the icosidodecahedron. These are the only semi-regular polyhedra that are *quasi-regular*.
8. Show there exist semi-regular polyhedra with vertex configuration
 - (a) 3.6.6
 - (b) 3.8.8
 - (c) 4.6.6
 - (d) 3.10.10
 - (e) 5.6.6
 - (f) 3.4.4.4 (the rhombicuboctahedron)
 - (g) 3.4.5.4 (the rhombicosidodecahedron)
 - (h) 4.6.8 (the great rhombicuboctahedron)
 - (i) 4.6.10 (the great rhombicosidodecahedron)
 - (j) 4.3.3.3.3 (the snub hexahedron)
 - (k) 5.3.3.3.3 (the snub dodecahedron)
 - (l) $n.4.4$
 - (m) $n.3.3.3$

The semi-regular polyhedra that are neither prisms nor antiprisms are known as the Archimedean solids. (It is unknown who discovered them, but it is believed that Archimedes wrote about them.)

9. For each of the above polyhedra, find the number of vertices, edges and faces.
10. Show that there are no other semi-regular polyhedra.

Relationships between Platonic solids

11. Show that one can pick four vertices of a cube that form a tetrahedron.

12. Show that one can pick four faces of an octahedron that form a tetrahedron.
13. Show that there exist cross-sections of the cube and octahedron that are regular hexagons.
14. Show that one can find eight points on the edges of an octahedron that form a cube.
15. Show that one can find six points on the edges of a cube that form an octahedron.
16. Show that one can pick eight vertices of a dodecahedron that form a cube.
17. Show that one can pick eight faces of an icosahedron that form an octahedron.
18. Show that one can find twelve points on the edges of an octahedron that form an icosahedron.
19. Show that there exists twelve planes tangent to the edges of a cube that form a dodecahedron.
20. Explore relationships among the above statements.

Questions to ponder

21. What are the regular and semi-regular tessellations of the plane? What are the regular and semi-regular tessellations of the hyperbolic plane?
22. How does one calculate the volumes of the above polyhedra?
23. What are the generalizations of regular polyhedra in higher dimensions?
24. Why doesn't there exist a semi-regular polyhedra with vertex configuration 3.7.7? 3.4.3.5? 5.5.6?
25. How do you explain that semi-regular polyhedra exist when there is no combinatorial obstruction?

References:

Wikipedia: Regular polytope, List of regular polytopes, Symmetrical polyhedra, Archimedean solids
 Prasadov, V. V., Sharygin, I. F.; Problems in Plane and Solid Geometry, Vol. 2, Chapter 9.
 Coxeter, H. S. M.; Regular Polytopes, (Methuen and Co., 1948).