Combinatorial Games I

Berkeley Math Circle, Intermediate Level I
April 14, 2015

In this session we will play some *combinatorial games*, where two players take turns making moves until one of them reaches a winning position. For all of the games below, first turn to your neighbor and play! After a few rounds, discuss who won and why. Try to find a winning strategy from a given starting position if this exists, or show that the second player must win.

1. **A subtraction game.** Starting with a pile of n chips (here n is a positive integer), two players alternate taking one to four chips. The player who removes the last chip wins.

   Play this game with your neighbor for different values of n! Try $n = 3$, $n = 4$, $n = 5$, $n = 6$, $n = 7$, $n = 10$, $n = 11$, $n = 28$, and $n = 40$. Which player has a winning strategy in these starting situations, the first player or the second? Can you determine the general winning strategy? What if the players are allowed to take one to five chips? How about one to seven chips?

2. **A subtraction game with different subtraction sets.** Again start with a pile of n chips, with two players alternating to take some number of chips. What happens if each player can only take 1 or 3 chips at each step? Who has a winning strategy for a given n? What if each player can only take 1 or 4 chips? How about taking 1, 2, or 4 chips?

3. **A subtraction game with two piles.** Suppose that now there are two piles of chips. The two players alternate taking turns, and each player can take either one to four chips from the first pile or one to five chips from the second pile. Who wins the game if initially there were 9 chips in the first pile and 9 in the second pile? What about $(9, 10)$ initially? And $(10, 10)$? Can you determine who has a winning strategy with initial condition (n, m)?

4. **Nim.** Now there are several piles, each containing finitely many chips. Two players alternate taking turns, and in each turn a player can remove any positive number of chips from a single pile. The winner is the player who removes the last chip.

   Play with your neighbor! Try several initial conditions with different number of piles. For instance, you can try two piles with initial chip counts $(3, 4)$ or $(4, 4)$. Try three piles with initial chip counts $(1, 2, 2)$, $(1, 2, 3)$, or $(2, 3, 5)$. How about four piles and $(1, 2, 3, 4)$?

   Can you determine who has a winning strategy for 2 piles with initial condition (n, m)? What is the winning strategy? How about for 3 piles? And 5 piles?

5. **Chomp.** This is a game that was invented in the 1970's by David Gale, who was a professor of mathematics here at the University of California, Berkeley.

   Chomp is played on a rectangular chocolate bar divided into smaller squares. Unfortunately, the bottom left square is poisoned. The players take turns, and at each step a player chooses an uneaten chocolate square and eats everything above and to the right of this square. The person who eats the poisoned square loses.

   Who wins the game for a 2×2 square? How about for a 2×3 square? And for a 3×3 square?

   Can you determine which player has a winning strategy in general, given a rectangular chocolate bar to start with?