4TH CLASS STARTER 9/23/14

If you're allowed to make only a single, planar (flat) cut in order to slice a cube of cheese into two pieces, what shapes can the resulting faces of the cheese-chunks have?

Can you get a pentagon?
A hexagon?
How about an octagon?

What shapes can result from cutting the cube into three pieces with two sequential planar cuts?

3RD CLASS STARTER 9/16/14

Part 1: (Mild) How many of the integers from one to forty are relatively prime to forty?

Definition: Two integers, a and b, are ***relatively prime*** if their greatest common factor is one.

Part 2: (Medium) How many of the integers from one to three-hundred are relatively prime to three-hundred?

Part 3: (Spicy) If p is some prime number, how many of the integers from 1 to p-hundred are relatively prime to p-hundred?

COMBINATORIAL GEOMETRY

Primes, Stars, and Hypercubes

PROBLEM SOLVING STRATEGIES

Three important strategies:

1) Using an organized approach
2) Presenting the solutions AND proving that no others exist
3) Testing small cases and reusing work from those cases

3RD CLASS STARTER 9/16/14

Part 1: (Mild) How many of the integers from one to forty are relatively prime to forty?

Definition: Two integers, a and b, are ***relatively prime*** if their greatest common factor is one.

Part 2: (Medium) How many of the integers from one to three-hundred are relatively prime to three-hundred?

Part 3: (Spicy) If p is some prime number, how many of the integers from 1 to p-hundred are relatively prime to p-hundred?

3RD CLASS STARTER 9/16/14

Part 1: (Mild) How many of the integers from one to forty are relatively prime to forty?

Definition: Two integers, a and b, are ***relatively prime*** if their greatest common factor is one.

1	X	3	X	X	X	7	X	9
X	11	X	13	X	X	X	17	X
19	X	21	X	23	X	X	X	27
X	29	X	31	X	33	X	X	X
37	X	39	X					

Answer: 16

3RD CLASS STARTER 9/16/14

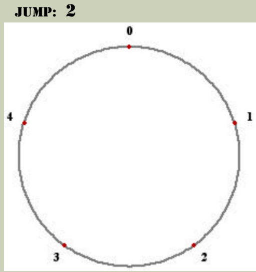
Part 2: (Medium) How many of the integers from one to three-hundred are relatively prime to three-hundred?

3RD CLASS STARTER 9/16/14

Part 3: (Spicy) If p is some prime number, how many of the integers from 1 to p-hundred are relatively prime to p-hundred?

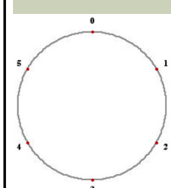
EULERIAN STARS

POINTS: 5
JUMP: 2

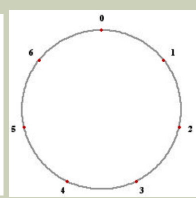


EULERIAN STARS

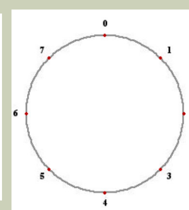
POINTS: 6
JUMP: 2



POINTS: 7
JUMP: 2

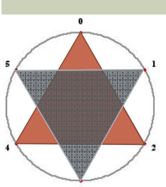


POINTS: 8
JUMP: 2

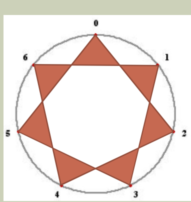


EULERIAN STARS

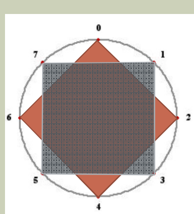
POINTS: 6
JUMP: 2



POINTS: 7
JUMP: 2

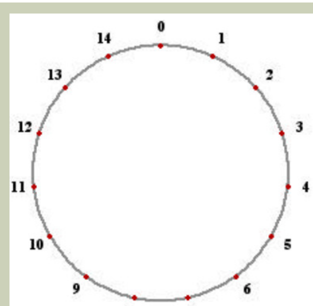


POINTS: 8
JUMP: 2



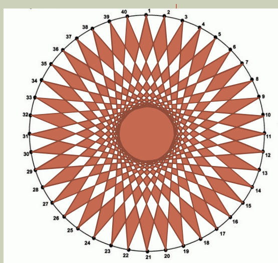
EULERIAN STARS

POINTS: 15
JUMP: ___?



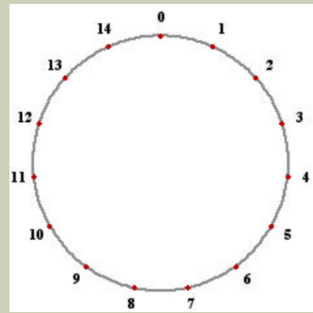
EULERIAN STAR COUNTING

- How many *different* 40-point Eulerian stars can be drawn?
- How many *different* 300-point Eulerian stars can be drawn?



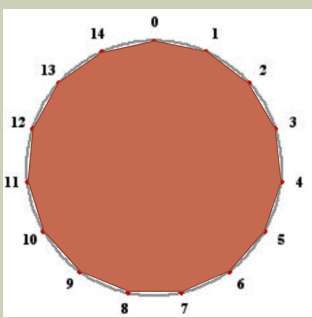
EULERIAN STARS

POINTS: 15
JUMP: ___?



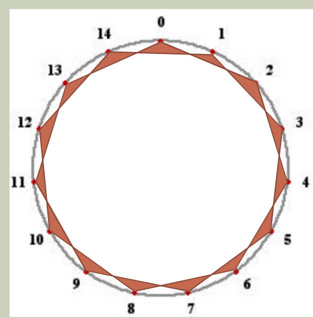
EULERIAN STARS

POINTS: 15
JUMP: 1



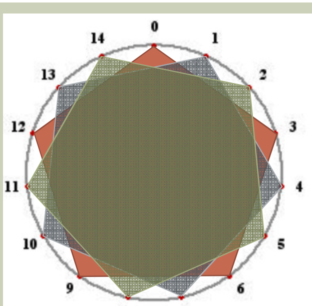
EULERIAN STARS

POINTS: 15
JUMP: 2



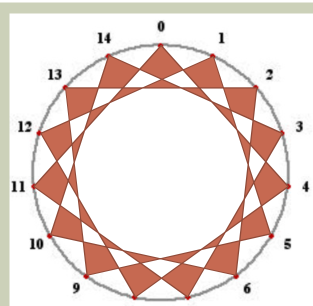
EULERIAN STARS

POINTS: 15
JUMP: 3



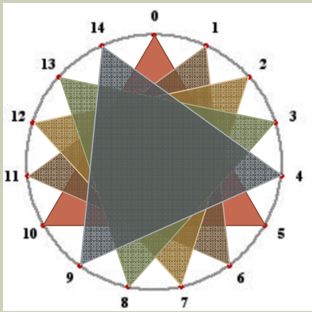
EULERIAN STARS

POINTS: 15
JUMP: 4



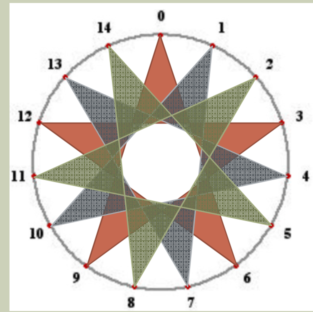
EULERIAN STARS

POINTS: 15
JUMP: 5



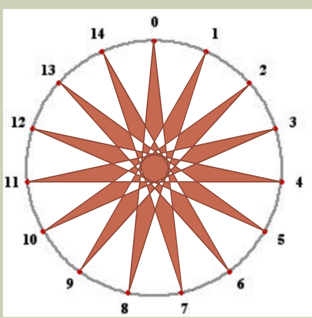
EULERIAN STARS

POINTS: 15
JUMP: 6



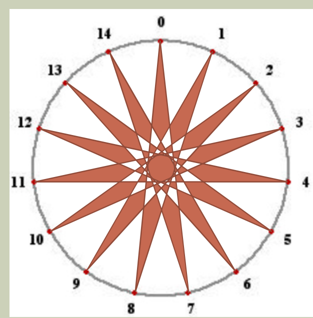
EULERIAN STARS

POINTS: 15
JUMP: 7



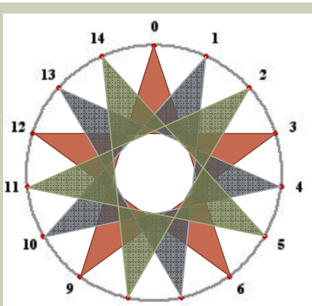
EULERIAN STARS

POINTS: 15
JUMP: 8



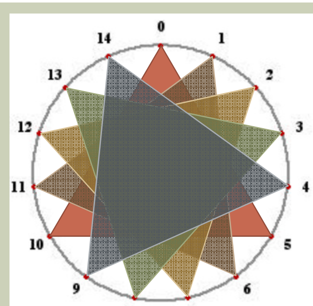
EULERIAN STARS

POINTS: 15
JUMP: 9



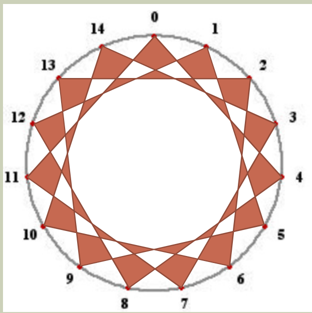
EULERIAN STARS

POINTS: 15
JUMP: 10



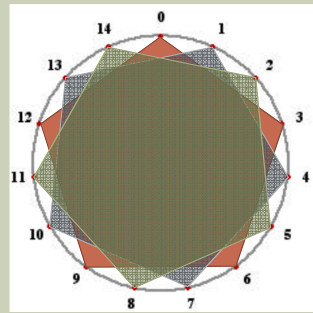
EULERIAN STARS

POINTS: 15
JUMP: 11



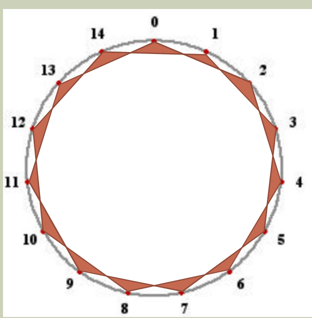
EULERIAN STARS

POINTS: 15
JUMP: 12



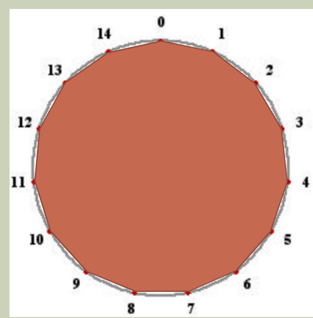
EULERIAN STARS

POINTS: 15
JUMP: 13



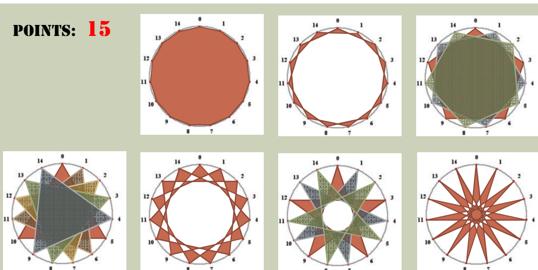
EULERIAN STARS

POINTS: 15
JUMP: 14



EULERIAN STARS

POINTS: 15



EULERIAN STAR COUNTING

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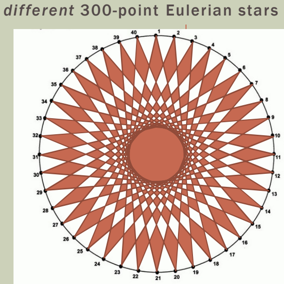
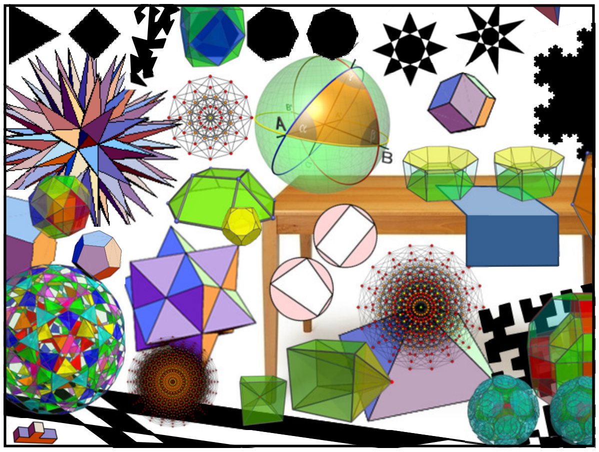
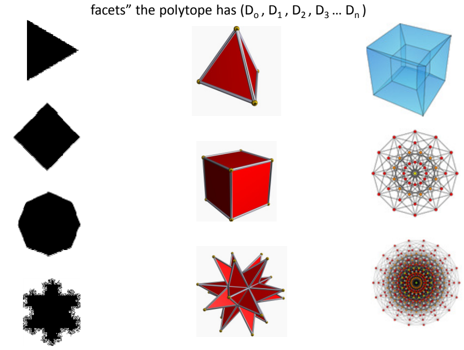




F-Vectors

A vector that describes how many of each "n-dimensional facets" the polytope has (D_o, D_1, D_2, D_3 … D_n)



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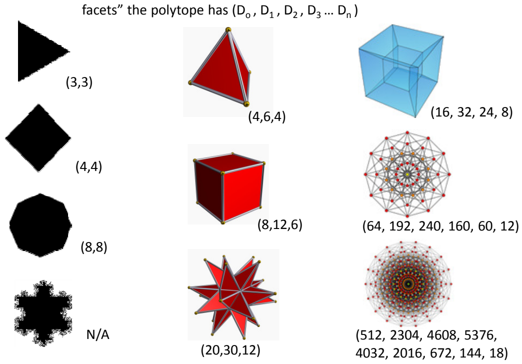
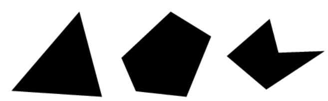
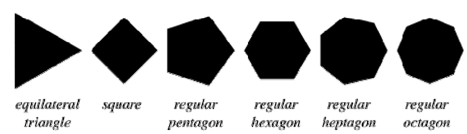


2D: Polygon



Definition 1: A closed, 2D figure with straight edges.
(Gellert et al. 2013, p. 137)

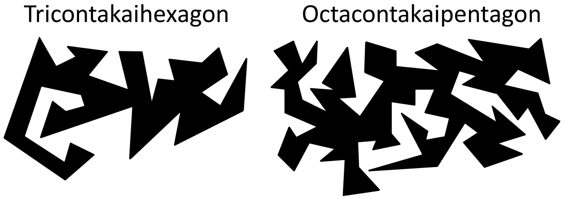
A polygon is ***regular*** if all of the sides and angles are equivalent:



Poly = many, Gonia = angle

Already, there are infinitely many!

Tricontakaihexagon Octacontakaipentagon



1 – hena
2 – di
3 – tri
4 – tetra
5 – penta

6 – hexa
7 – hepta
8 – octa
9 – ennea
10 – deca

+ – kai

x 10 – conta

"angeled shape" – gon

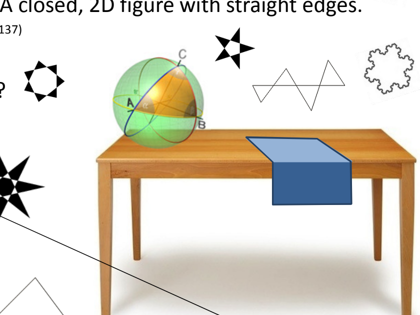
__contakai___gon

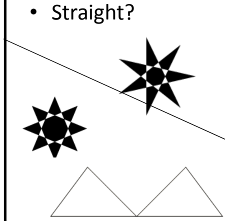
Exceptions:
Triangle
Quadrilateral
20 – icosa

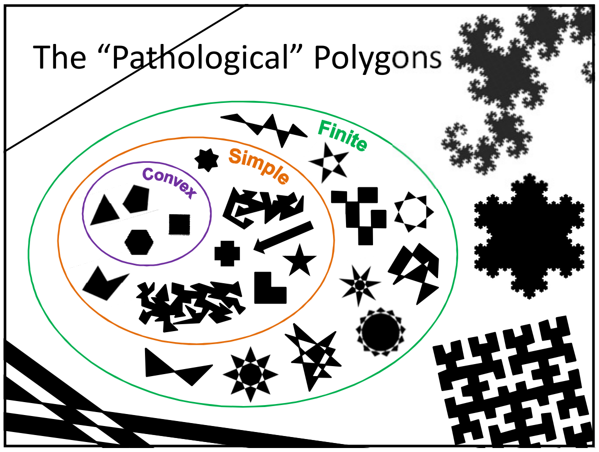
Polygon

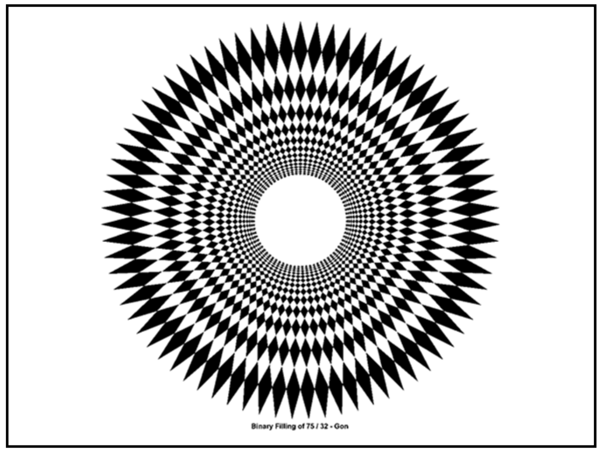
Definition 1: A closed, 2D figure with straight edges.
(Gellert et al. 2013, p. 137)

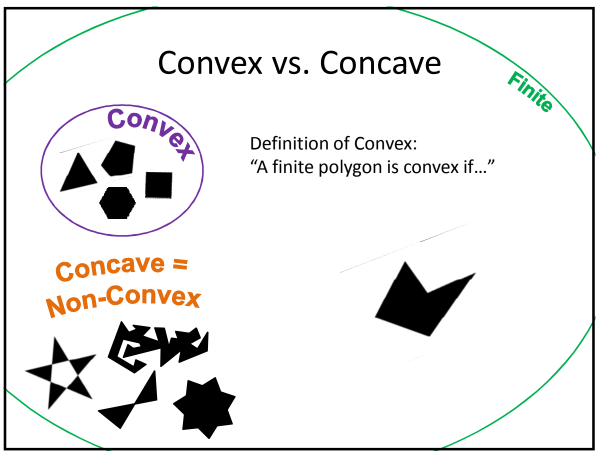
- Closed?
- 2D/planar?
- Straight?

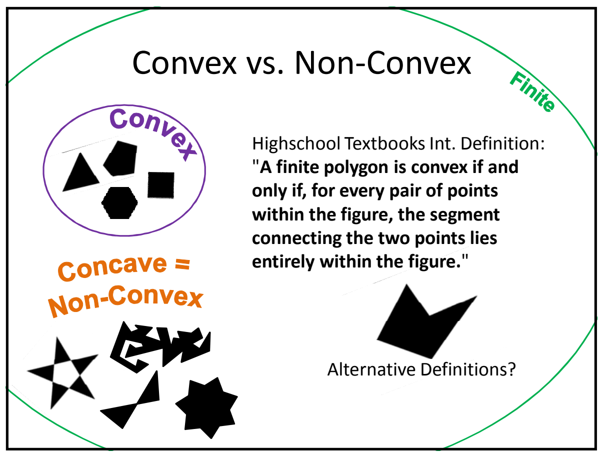












WHICH OF THE FOLLOWING DEFINITIONS ARE EQUIVALENT?

HTbI: **"A (finite) polygon is convex if and only if for every pair of points within the polygon, the line segment connecting the two points lies entirely within the polygon."**

Alternatives: A polygon is convex if and only if….

a. all diagonals lie in the interior of the polygon.
b. there is no straight line that intersects its edge at four or more points.
c. the perimeter is larger than the length of the longest diagonal.
d. everydiagonalislongerthaneveryside.
e. the perimeter of the polygon is the shortest path that encloses the entire shape.
f. the largest interior angle is(are) adjacent to the longest side(s).
g. none of the lines that contain the sides of the polygon pass through its interior.
h. every interior angle is less than 180°.
i. a circle can be inscribed within it which touches every edge.

Solution:

a, b, e, g, and h are all equivalent to:

"A (finite) polygon is convex if and only if for every pair of points within the polygon, the line segment connecting the two points lies entirely within the polygon."

Inequivalent definitions:

A polygon is convex <u>if and only if</u>….

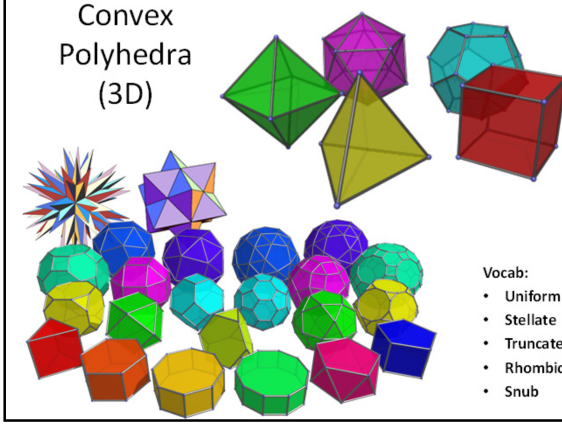
c. the perimeter is larger than the length of the longest diagonal.
d. every diagonal is longer than every side.
f. the largest interior angle is(are) adjacent to the longest side(s).
i. a circle can be inscribed within it which touches every edge.

“Convex” in 3+ Dimensions

An ***n-dimensional polytope*** is a finite region of n-dimensional space enclosed by a finite number of n-1 dimensional hyperplanes.

A n-dimensional polytope is ***convex*** if, for every pair of points within the figure, the segment connecting the two points lies entirely within the figure.

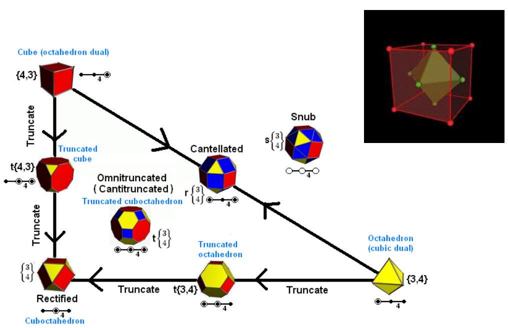
Convex Polyhedra (3D)



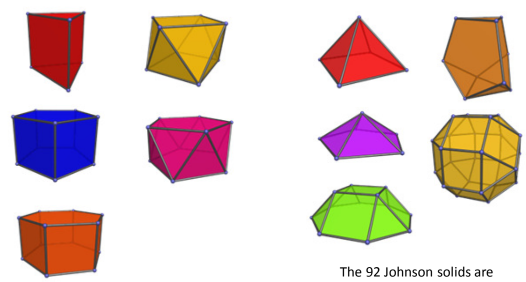
Vocab:

- Uniform
- Stellate
- Truncate
- Rhombic
- Snub

Duals (octahedron/cube)

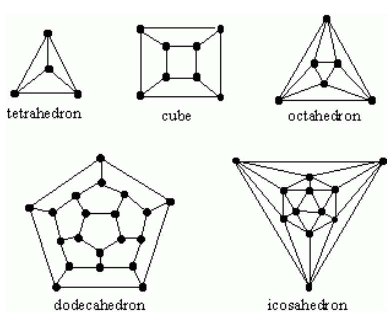


Prisms and Antiprisms/ The Johnson Solids

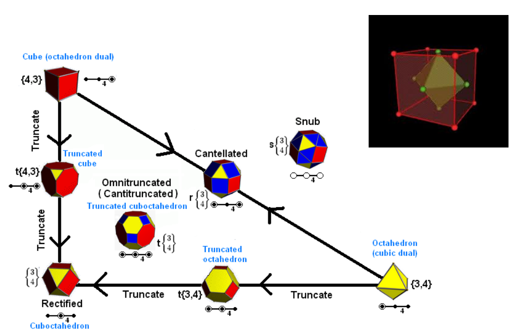


The 92 Johnson solids are convex and have regular polygons for their faces, but they are non-uniform

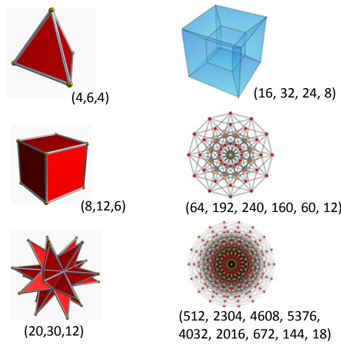
Polygon Net



Duals (octahedron/cube)

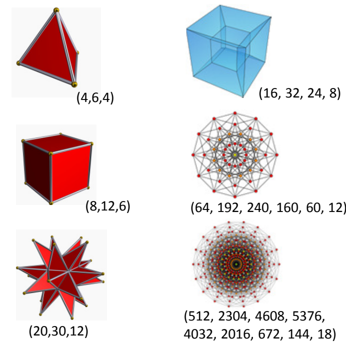


F-Vector: Euler's Formula



F-Vector: Euler's Formula

V-E+F = 2



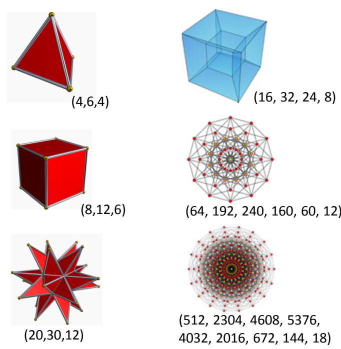
F-Vector: Euler's Formula

V-E+F = 2

Generalized:

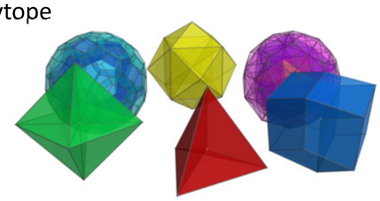
$\Sigma(-1)^i f_i = 0$

Make every other quantity in the f-vector negative, then add them together.

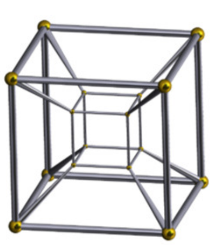


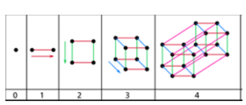
Regular, Convex Polytopes (N-dimensional)

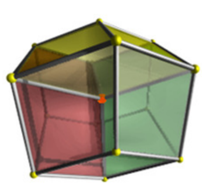
- Simplex
- Hypercube
- Cross-Polytope



The 4D Hypercube







HYPERCUBE F-VECTORS

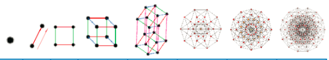
N-Dimensional Hypercubes

	0	1	2	3	4	5	6	7
V (d0)								
E (d1)								
F (d2)								
d3								
d4								
d5								
d6								

General Formula:

HYPERCUBE F-VECTORS

N-Dimensional Hypercubes

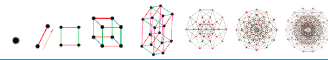


	0	1	2	3	4	5	6	7
V (d0)	1	2	4	8	16			
E (d1)	-	1	4	12	32			
F (d2)	-	-	1	6	24			
d3	-	-	-	1	8			
d4	-	-	-	-	1			
d5	-	-	-	-	-			
d6	-	-	-	-	-			

General Formula:

HYPERCUBE F-VECTORS

N-Dimensional Hypercubes

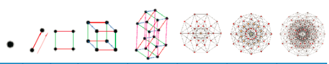


	0	1	2	3	4	5	6	7
V (d0)	1	2	4	8	16	32	64	128
E (d1)	-	1	4	12	32			
F (d2)	-	-	1	6	24			
d3	-	-	-	1	8			
d4	-	-	-	-	1			
d5	-	-	-	-	-			
d6	-	-	-	-	-			

General Formula:

HYPERCUBE F-VECTORS

N-Dimensional Hypercubes

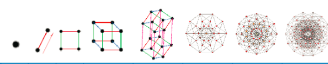


	0	1	2	3	4	5	6	7
V (d0)	1	2	4	8	16	32	64	128
E (d1)	-	1	4	12	32	80	192	452
F (d2)	-	-	1	6	24			
d3	-	-	-	1	8			
d4	-	-	-	-	1			
d5	-	-	-	-	-			
d6	-	-	-	-	-			

General Formula:

HYPERCUBE F-VECTORS

N-Dimensional Hypercubes

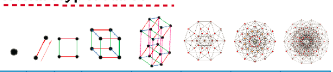


	0	1	2	3	4	5	6	7
V (d0)	1	2	4	8	16	32	64	128
E (d1)	-	1	4	12	32	80	192	452
F (d2)	-	-	1	6	24	80	240	672
d3	-	-	-	1	8			
d4	-	-	-	-	1			
d5	-	-	-	-	-			
d6	-	-	-	-	-			

General Formula:

HYPERCUBE F-VECTORS

N-Dimensional Hypercubes

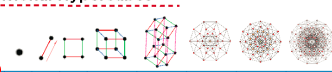


	0	1	2	3	4	5	6	7
V (d0)	1	2	4	8	16	32	64	128
E (d1)	-	1	4	12	32	80	192	452
F (d2)	-	-	1	6	24	80	240	672
d3	-	-	-	1	8	40	160	560
d4	-	-	-	-	1	10	60	280
d5	-	-	-	-	-	1	12	84
d6	-	-	-	-	-	-	1	14

General Formula:

HYPERCUBE F-VECTORS

N-Dimensional Hypercubes



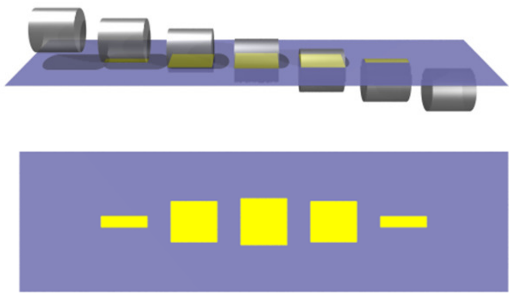
m \ n	0	1	2	3	4	5	6	7
V (d0)	1	2	4	8	16	32	64	128
E (d1)	-	1	4	12	32	80	192	452
F (d2)	-	-	1	6	24	80	240	672
d3	-	-	-	1	8	40	160	560
d4	-	-	-	-	1	10	60	280
d5	-	-	-	-	-	1	12	84
d6	-	-	-	-	-	-	1	14

General Formula:

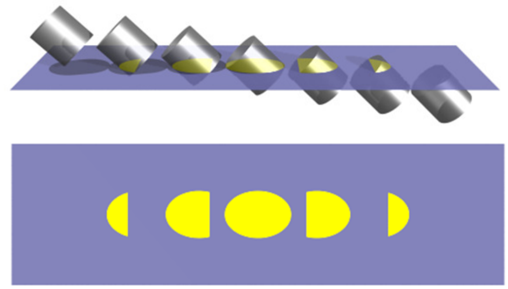
$$\frac{2^{(n-m)}\, n!}{m!\,(n-m)!}$$

Note:
0! = 1

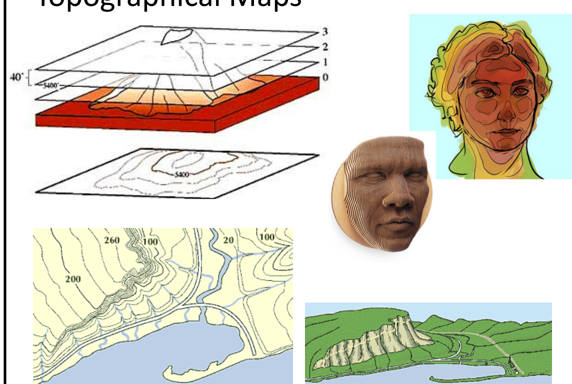
Cross-Sections



Cross-Sections



Topographical Maps



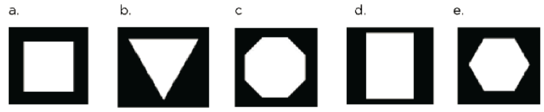
PLANAR CROSS-SECTIONS

Which of the following polygons are cross-sections of a tetrahedron?

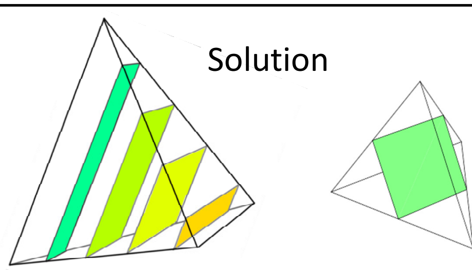
a. b. c. d. e.

Which of the following polygons are cross-sections of a tetrahedron?

a. b. c d. e.

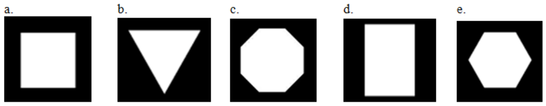


Solution

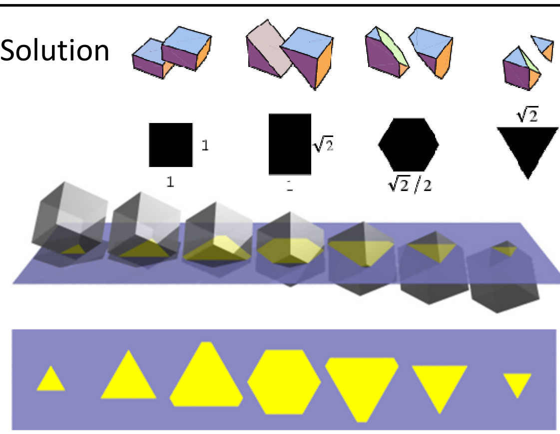


Which of the following polygons are cross-sections of a tetrahedron?

a. b. c. d. e.

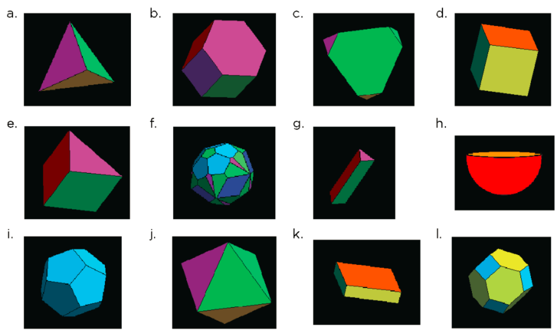


Solution



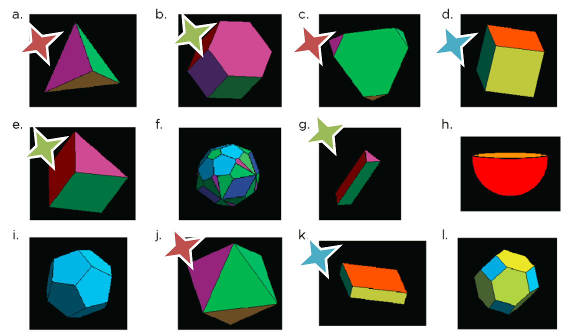
HYPERPLANAR CROSS-SECTIONS

Which of the following polyhedra are cross-sections of a hypercube?



Solutions

Which of the following polyhedra are cross-sections of a hypercube?



Vertex-First Cross-Section of a 4D Hypercube

Cube (3D)

Hypercube (4D)

Advanced Reading

- Generalizing Euler's Equation to 4 Dimensions
  - http://www.math.hmc.edu/~su/pcmi/projects/polytope_outreach/outreach.pdf
- Creating Flipbook Polyhedra
  - http://archive.bridgesmathart.org/2013/bridges2013-619.pdf
- The Pentatope (4D tetrahedron)
  - http://eusebeia.dyndns.org/4d/5-cell