FREEDOM FOR THE CLONES

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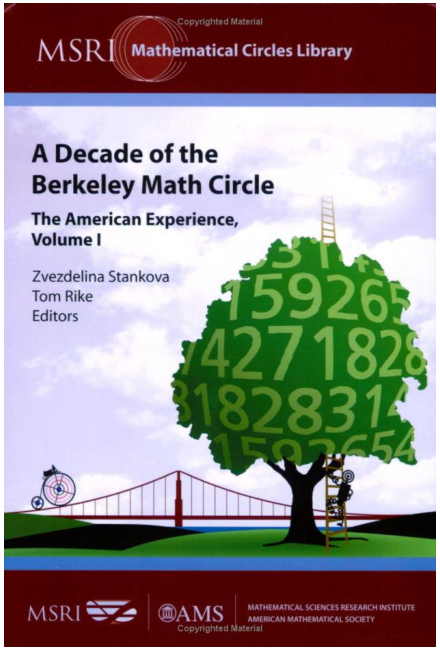

Session 10

Stomp. Games with Invariants

BASED ON SAM VANDERVELDE'S SESSION

SNEAK PREVIEW. The reader will encounter games and puzzles with exotic names like *Gopher Gun*, *Escape of the Clones*, and *Pointless Machine*, which are connected by the fundamental problem solving technique of *invariants*. When organized and led suitably, the material here can provide a highly interactive environment where group work, individual investigation, and class discussions flow seamlessly from one to the other; and indeed the *live BMC Stomp* session involved with equal enthusiasm everyone from the youngest middle schoolers to the high school seniors. Although the concept is not new, the game of *Stomp* was created specifically for the BMC session, along with all the related problems. Stomp has appeared on a few other occasions – most notably at the EPGY Summer Programs '05–06 [**27**] and the Julia Robinson Math Festival '07 [**77**], while Gopher Gun originated at Mathcamp '05 [**14**].

5. Escape of the Clones

This is a version of a famous puzzle attributed originally to Maksim Kontsevich, which appeared in the Tournament of the Towns and in the Russian journal *Kvant* in 1981 (cf. [**54, 50**]). Its solution will require the creation of invariants with *infinite series*.

5.1. The set-up of the game. Consider the first quadrant in the Cartesian plane divided into unit squares by horizontal and vertical lines at the positive integers. Place 3 dots (*clones*) in the shape of an L-tromino in the bottom left-most squares, and draw a "barbed wire fence" enclosing the dots and their 3 respective squares: this is the orange fence in Figure 8.

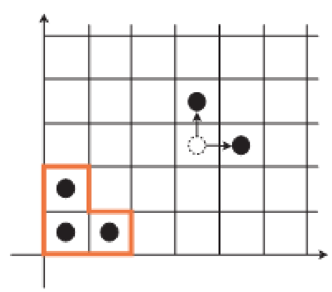

FIGURE 8. Escape of the Clones

5.2. The rules of the game. At each step you can erase a dot and replace it with two copies in adjacent squares, one directly above and the other directly to the right, as long as those squares are currently unoccupied. In other words, when a clone disappears, it sprouts two more clones above and to the right of it. Notice that this is a Stomp-like game

- whose board is the infinite first quadrant and
- whose "footprint" is an L-tromino allowed to be placed only in the standard orientation of the English letter L, and only when the corner square of L covers a clone while the other two squares of L land on clone-empty spots.

5.3. Freedom for the clones!

Problem 14 (Advanced). Prove that it is impossible to free all clones from the prison.

Although the problem setting is elementary enough for anyone to play and enjoy the game, the actual solution is hard to come up with and requires knowledge of a useful summation formula. That's great: while playing games, we will learn some algebra too!

5.4. In search of the invariant. What could be the invariant preventing us from freeing all 3 prisoners? You should try parity, the coloring technique, and any other previous ideas, but soon enough you will realize that the problem is too complicated to succumb to these methods. We need a different, more powerful approach here:

PST 78. *Assign a suitable number to each square* to create an invariant.

What could these suitable numbers be? Let us agree to label a square with the coordinates (a, b) of its bottom left corner. Thus, the 3 initially occupied squares are labeled by $(0, 0)$, $(1, 0)$, and $(0, 1)$. If we assign, for instance, the number 1 to square $(0, 0)$, it makes sense to assign $1/2$ to each of squares $(1, 0)$ and $(0, 1)$ so that the sum of assigned numbers to occupied squares before a move and after a move remains constant: $1 = 1/2 + 1/2$. OK, but then we are more or less forced to assign $1/4$ to each of squares $(2, 0)$, $(1, 1)$, and $(0, 2)$ so that this same reasoning works when a clone in $(1, 0)$ or $(0, 1)$ sprouts two more clones: $1/2 = 1/4 + 1/4$.

Thus, every time we move to the right or up, the assigned numbers get halved! Going around the board in this manner, we soon discover an exact formula for all the desired numbers:

Exercise 16. To every square (x, y) assign the number $\frac{1}{2^{x+y}}$. Show that when a move is applied to a clone anywhere, the sums of the numbers assigned to occupied squares before and after the move are equal.

We have found our invariant: the sum of the numbers in occupied squares stays constant throughout the game! How can we capitalize on this invariant?

5.5. Arguing by contradiction. Suppose that we *can* free all 3 clones from the prison. Along the way, they will have sprouted several clones outside the prison. Call the sum of the numbers in occupied squares at the end of the game S_{out}. Note that our initial invariant sum for the clones in the prison is simply $S_{\text{in}} = 1 + \frac{1}{2} + \frac{1}{2} = 2$; so we must have $S_{\text{out}} = 2$. But could this happen? Well, whatever happens, S_{out} *cannot* be more than the sum of all numbers assigned to out-of-prison squares.

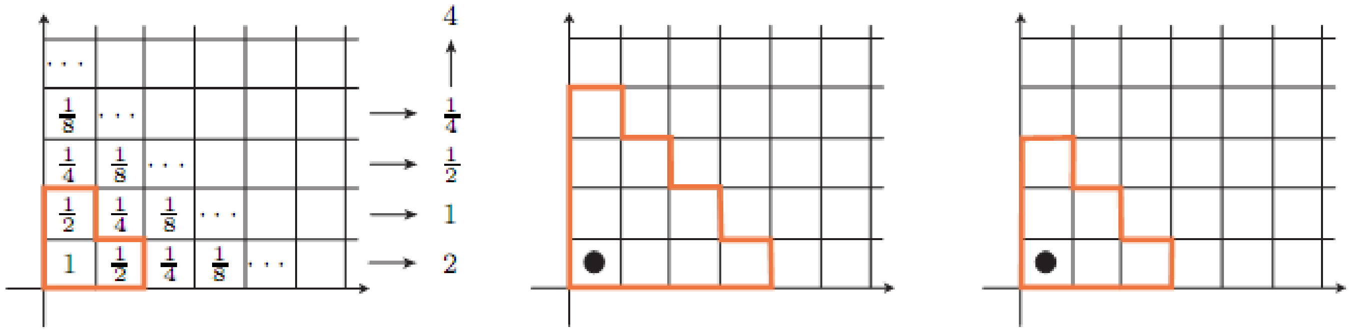

FIGURE 9. Assignment of numbers and Problem 16

5.6. Algebra detour. For starters, let's add up *all* numbers on the infinite board in an orderly fashion. For example, on row 1 we will have

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2. \tag{3}$$

If the reader wonders how we came up so fast with the answer for this sum, note that a much more general formula holds:

Problem 15. For any (real) numbers a and r, $-1 < r < 1$, show that

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots = \frac{a}{1-r}.$$

The sum is called a *geometric series* with *initial term* a and *ratio* r. We leave the proof of this formula for the Hints section, but we do encourage the reader to try to prove it from scratch: it won't be a trivial exercise.[6]

In the case of row 1 (cf. Fig. 9a), the initial term is $a = 1$, and the ratio is $r = \frac{1}{2}$. Now, to see what goes on in row 2, note that each number there is half of the number directly below it, i.e., the sum in row 2 will be half of the sum in row 1. Similarly, the sum in row 3 will be half of the sum in row 2, and so on. In general, in the i^{th} row:

$$\frac{1}{2^{i-1}} + \frac{1}{2^i} + \frac{1}{2^{i+1}} \cdots = \frac{1}{2^{i-1}}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) = \frac{1}{2^{i-1}} \cdot 2 = \frac{1}{2^{i-2}}.$$

Adding up the sums in all rows yields the sum of *all* numbers on the board:

$$2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2 + 2 = 4.$$

All right, we subtract the "in-prison" numbers, $1 + \frac{1}{2} + \frac{1}{2} = 2$, to obtain the maximal value that S_{out} can ever be: $S_{\text{out}} \leq 4 - 2 = 2$.

[6] This formula appears in just about any course in calculus when discussing *sequences* and *series*. We shall dedicate a session to related topics in a future volume.

5.7. The finishing touch: this is a finite game! Recall that, even though we are playing on an infinite board, the prisoners are supposed to be freed in *finitely* many moves. Thus, at least one outside-of-prison square is clone-free hence the out-of-prison sum is strictly less than 2: $S_{\text{out}} < 2$. This contradicts the fact that $S_{\text{out}} = S_{\text{in}} = 2$ and proves that the three clones cannot all be freed from prison. □

5.8. Pushing on: generalizations. What is the real story behind this problem? Why did the method of using a summation invariant work so nicely? If we slightly change the initial set-up, would it still be impossible to free the clones?

Problem 16. A single clone is placed in square $(0, 0)$, and the prison encloses

(a) the 10 squares (i, j) with $i + j \leq 3$ (Kontsevich [**54**]; cf. Fig. 9b);

(b) the 6 squares (i, j) with $i + j \leq 2$ (Khodulev [**50**]; cf. Fig. 9c).

Show that there will always be at least one clone in the prison.

Hint: Part (a) is like Problem 14, but part (b) requires another insight. ◊

Further Reading:

A delightful discussion and generalization of the above problems is presented by Chung, Graham, Morrison, and Odlyzko in a 1995 piece in the *Monthly* (cf. [**15**]). Starting with one initial clone in $(0, 0)$, the articles describes all "inescapable prison shapes," and relates the problem to a new "Ramanujan-like" *continued fraction*.[7]

15. F. Chung, R. Graham, J. Morrison, and A. Odlyzko, *Pebbling a Chessboard*, Amer. Math. Monthly **102** (1995), no. 2, 113–123.

    http://www.math.ucsd.edu/~fan/mypaps/fanpap/150chess.pdf

50. A. Khodulev, *Pebble Spreading*, Kvant (1982), 28–31, 55.

54. M. Kontsevich, *Problem M715*, Kvant (1981), 21.

[7] The beginning of [**15**] can be readily followed by anyone. However, as the reader advances into the article, more maturity and experience will be required. For example, familiarity with the notions of recursive sequences, summation techniques, partial derivatives, continued fractions, and asymptotic manipulations will be needed. The advanced reader may consider learning the necessary background for the article as a long-term project.

For the Die-Hards:

The Game (Conway Checkers): On an infinite square grid, a horizontal line of the grid is drawn. You initially place checkers below the line: as many as you wish, but no more than 1 checker per square. Then you may take a checker and jump it over a checker that is adjacent to it (in any of the four directions) into the square immediately beyond, if that square is vacant. In the process you remove the square that has been jumped over. You may continue jumping checkers as long as there are two checkers adjacent to each other somewhere. The goal is the get a checker as far above the drawn horizontal line as possible. What is the highest row that can be reached and why?

Note: The solution involves some creativity in constructing examples to reach the first several rows, and a lot of creativity in using infinite series and the golden ratio in showing that one cannot reach any further row.

6. Hints and Solutions to Selected Problems

Problem 15. Assume for the moment that the desired sum exists, i.e.,

$$a + ar + ar^2 + \cdots + ar^n + \cdots = S \tag{5}$$

for some number S. Multiplying by the common ratio r gives

$$ar + ar^2 + ar^3 + \cdots + ar^{n+1} + \cdots = rS. \tag{6}$$

Now subtracting the two equations (5) and (6) yields

$$a = (1-r)S \Rightarrow S = \frac{a}{1-r}.$$

Thus when $a = 1$ and $r = 1/2$, we obtain the special case of the row sum in the Clones problem: $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-1/2} = 2$.

To prove that the sum S actually exists is a completely different ball-game, which we leave to the reader to ponder over. You will encounter a complete proof of this in a calculus course, where limits are introduced. ◊

Problem 16. Use the same assignment of 2^{-i-j} to squares (i, j), and compare sums $S_{\text{in}} = 1$ and S_{out}, which are supposed to be equal. In part (a), a contradiction is achieved via the inequalities

$$S_{\text{out}} < 4 - 1 - 2 \cdot \tfrac{1}{2} - 3 \cdot \tfrac{1}{4} - 4 \cdot \tfrac{1}{8} = \tfrac{3}{4} < 1.$$ ↯

The same adding technique produces in part (b)

$$S_{\text{out}} < 4 - 1 - 2 \cdot \tfrac{1}{2} - 3 \cdot \tfrac{1}{4} = \tfrac{5}{4} > 1,$$

which does not provide a contradiction. A more refined argument for S_{out} is needed. Note that in the first row and first column of the board there will always be exactly one clone (why?); hence this row and column will contribute to S_{out} at most $1/8$ each. Add up the rest of the board, and find a smaller upper bound for S_{out}: ↯

$$S_{\text{out}} < 4 - (2 - \tfrac{1}{8}) - (2 - \tfrac{1}{8}) + 1 - \tfrac{1}{4} = 1.$$ ◊