Mathematical Induction

Berkeley Math Circle

13 January 2015

Ted Courant

tcourant@bentleyschool.net

Patterns and Expressions

1. Suppose you have an unlimited number of 3 and 5 cent stamps.
2. Prove that you can make any amount of postage 8 cents or more.
3. What is the formula for the n^{th} odd number? The $(n + 1)^{\text{st}}$ odd number?
4. Find and prove a formula for the sum $1 + 2 + 3 + \cdots + n$. These are called *triangular* numbers.
5. Draw the first 10 rows of Pascal's Triangle. Do you see the triangular numbers?
6. Prove the Hockey Stick theorem (for Pascal's Triangle).
7. Use the Hockey Stick Theorem to prove…

   a) … that $1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n = \frac{n \cdot (n+1) \cdot (n+2)}{3}$

   b) … that $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + (n-1) \cdot n \cdot (n+1) = \frac{n \cdot (n+1) \cdot (n+2) \cdot (n+3)}{4}$

Mathematical Induction

Mathematical induction is used to prove that a fact is true for all (natural number) values of n.

•Proving *the fact* works for the initial term (or terms) of the sequence is the *anchor*.

•Proving that if *the fact* is true for $n - 1$, then it is true for n is the inductive (or recursive) step.

•If you prove both, you are done.

8. Use induction to prove that $6^n - 1$ is always divisible by 5.
9. Prove that $1 + 2 + 4 + 8 + \cdots + 2^{n-1} = 2^n - 1$
10. Prove that it is possible to cut up n squares and rearrange the pieces into a single square.
11. Prove that 99^n always ends in 01 or 99.
12. Prove that the sum of three consecutive cubes is always divisible by 9, e.g. $9^3 + 10^3 + 11^3$ is a multiple of 9.
13. Find and prove a formula for the sum of the first n odd numbers.
14. What is the greatest number of pieces you can get by making n straight cuts through a circular pizza?
15. Prove that $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$
16. Use induction to prove that $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$.

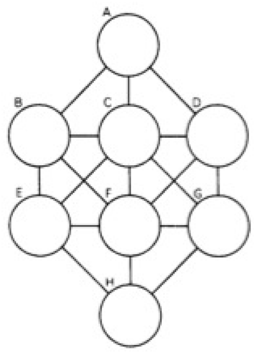
    (Try to find another simple way to establish this inequality).
17. Prove that $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} < 2$.

    (Hint: we pass from $1 + \frac{1}{2}$ to to $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ by dividing by two and adding one)
18. Use induction to prove that $n^2 + (n+1)^2 + (n+2)^2$ is always divisible by 9.
19. Use Induction to prove…

    a) … that $1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n = \frac{n \cdot (n+1) \cdot (n+2)}{3}$

    b) … that $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + (n-1) \cdot n \cdot (n+1) = \frac{n \cdot (n+1) \cdot (n+2) \cdot (n+3)}{4}$

Warm-up Questions



Place the digits 1–8 into the circles so that no neighbors are connected by the given line segments.

What is the least possible value of the smallest of 99 consecutive positive integers whose sum is *a* perfect cube?
(2014 California Math League, 14 January, 2013)

Prove the Two-Color Map Theorem: if non-intersecting curve extend to the boundary of the region, then two colors suffice to color the map so that no adjacent regions (those sharing a boundary curve) have the same color.