9. Area of a Triangle

Problem 1. Draw a rectangle on your graph paper. Calculate its area. Describe a general formula for the area of a rectangle.

The area of a rectangle is calculated by:

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Theorem 1. The area of a rectangle is the product of two adjacent sides:

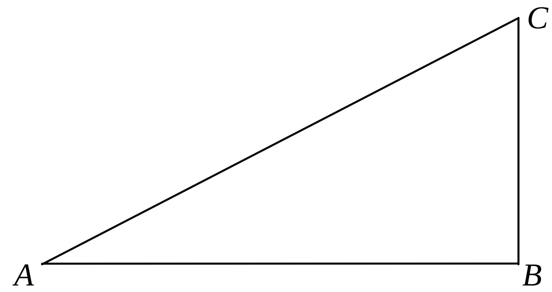
Area of a rectangle = length x width

= base x height.

Problem 2. Now draw a *right* triangle on your graph paper? How much is its area? Conjecture a formula that should work for *any right* triangle.

Hint 1. Can you somehow reduce to the previous situation with the rectangle? Isn't every right triangle part of a rectangle?

Definition 1. In a right triangle *ABC* the two sides that make the right angle are called the *legs*, and the side opposite the right angle is called the *hypotenuse* of the triangle.



Problem 3. Complete the right triangle *ABC* above to a rectangle: use the hypotenuse as the *diagonal* of the rectangle and draw some perpendiculars to the legs make the rectangle. What part (how much) of the rectangle is the triangle?

The area of the right triangle *ABC* is equals:

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Lemma 1. The area of a right triangle is half of the product of the two legs:

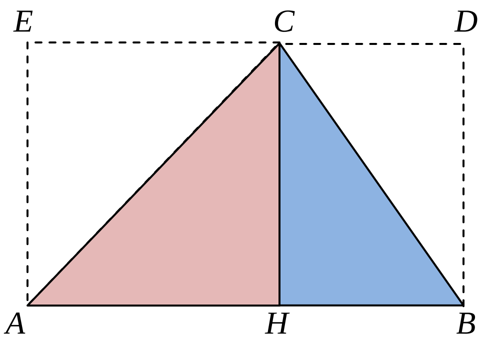
Area of a right triangle equals:

– (leg x leg) = – base x height).

Problem 4. Draw on your graph paper an *acute* triangle *ABC* (let *AB* be horizontal). Calculate the area of the triangle *ABC*.

Hint 2. Reduce to the previous situation with the right rectangles: drop the altitude *CH* from *C* to side *AB*. How many right triangles do you see? Color them differently.

Hint 3. Complete the right triangles *AHC* and *BHC* to rectangles. Do you see a big rectangle in which the original triangle *ABC* is *inscribed*? What is the area of this big rectangle? What part (how much) of the big rectangle is triangle *ABC*?



Proof. The *acute* triangle *ABC* is half of the rectangle *ABDE*. Since the base of the rectangle is the side *AB* and the height of the rectangle is the altitude *CH*, the area of the rectangle is *AB* x *CH*. Hence the area of the triangle is half of that.

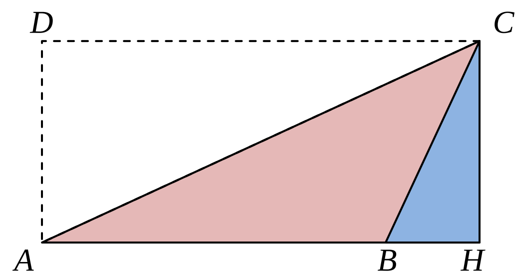
Lemma 2. The area of an *acute* triangle *ABC* is half of its base x height:

Area of an acute triangle equals:

– side x altitude) = – (base x height)

Problem 5. Final challenge: draw on your graph paper an obtuse triangle *ABC* with side *AB* horizontal and an obtuse angle at *B*. Drop the height *CH* from *C* to (line) *AB*. Complete the right triangle *AHC* to a rectangle *AHCD*. Can you use this picture to find the area of the original triangle *ABC*? Write down your conjecture.

Definition 2. Let *CH* be the altitude in triangle *ABC*. Point *H* is called the *foot* of the altitude, or the *foot* of the perpendicular from *C* to *AB*.



Proof. We already know how to find the areas of right triangles *AHC* and *BHC:* multiply their legs and divide by 2:

- Area of right *AHC* = – *AH* x *HC*
- Area of right *BHC* = – *BH* x *HC*

Since the area of triangle ABC is the difference of these areas, we obtain:

- Area of obtuse *BHC* =

  = – *AH* x *CH* - – *BH* x *CH* =

  = – (*AH-BH)* x *CH*= – *AB* x *CH.*

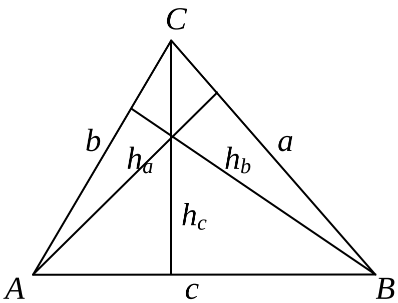
But this is again half of the base times the height? We are getting the same formula for any type of triangle!

Theorem 2. The area of *any* triangle *ABC* is half of its base x height:

Area of any triangle equals:

– side x altitude) = – (base x height).

Question 1. There is one formula for the area of a *rectangle*. How many formulas are there for the area of a *triangle*?



Standard Notation. Denote each side by the lower-case letter of the vertex opposite to that side: $AB = c$, $BC = a$, and $CA = b$, Denote also the altitude from a vertex by h with subscript the vertex's lower-case letter:

- the altitude from B to side AC is h_b,
- the altitude from A to side BC is h_a,
- the altitude from B to side AC is h_b.

Corollary 1. Each of the three altitudes in a triangle gives a formula for the area of the triangle. Thus, we have three formulas for the area of any triangle:

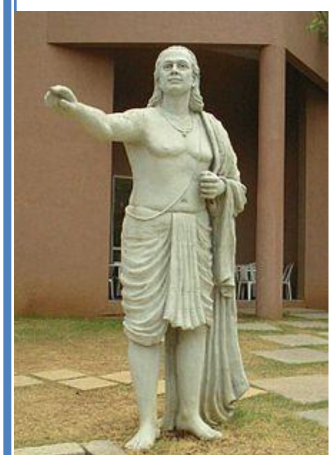
Area of any triangle equals:

$$= - a \text{ x } h_a = - b \text{ x } h_b = - c \text{ x } h_c$$

Problem 6. Use any of your previous triangles. Draw its three altitudes and calculate the area of the triangle in three different ways. Did you get exactly the same answer or approximately the same answer? How do you explain this? Are the three formulas for the area wrong?

Explanation. Most likely you got 3 close but not exactly the same answers. The difference in the answers is explained by the errors of drawing the altitudes and measuring the lengths of the sides and the altitudes. You got approximations, but not necessarily the exact area in all three cases. The formulas hold and they are correct!

Historical Facts. The computation of areas has a long history. The ancient *Egyptians*, *Babylonians*, and *Hindus* knew how to compute areas of simple geometric figures like triangles, squares, rectangles, parallelograms, and trapezoids. In 499 CE *Aryabhata*, a great mathematician-astronomer from the classical age of Indian mathematics and Indian astronomy, used the method of multiplying the base times the height and dividing by 2 in the Aryanhatiya. The Aryanhatiya is a Sanskrit astronomical treatise (written in verses!), is the *magnum opus* and only surviving work of Āryabhaṭa.



Statue of Aryabhata on the grounds of Inter-University Centre for Astronomy and Astrophysics in Pune, Maharashtra, India

He is known for explaining the lunar eclipse, and solar eclipse, rotation of earth on its axis, reflection of light by moon, sinusoidal functions, solution of single variable quadratic equation, value of π correct to 4 decimal places, circumference of the Earth to 99.8% accuracy.

Practical Facts. Although simple, our formula for the triangle's area is only useful if the height can be readily found, which is not always the case. For example, the surveyor of a triangular field might find it relatively easy to measure the length of each side, but relatively difficult to construct a 'height.' Other methods are used in practice, depending on what is known about the Δ

RECAP 1: Grouping Words

Split the 49 words into groups and make 4 true statements discussed in this lesson. Use all given clues.

adjacent angle. a any any area area area
base by
calculated
half height. hypotenuse
is is is is
legs.
of of of of of of opposite
product product
rectangle right right
sides.
taking times triangle triangle two two
The The The The the the the the the the

1. ______________________________________
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2. ______________________________________
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4. ______________________________________
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RECAP 2: New Vocabulary and Ideas

Check ALL correct answers. Explain your choice and provide details.

1. The *foot* of an altitude in a triangle:
   - I. Is where a perpendicular and a side of the triangle intersect.
   - II. Is the foot of a hill.
   - III. Is the intersection of the *diagonals* of a in a rectangle.
   - IV. Is the intersection of the altitudes in a triangle.
   - V. Is a point on the side of a triangle.
   - VI. Looks like the heel of your foot when you stand up.

2. The *area of a right* triangle:
   - I. Cannot be calculated unless you know the hypotenuse of the triangle.
   - II. Is half of the area of a suitable rectangle.
   - III. Was first calculated by Aryabhata.
   - IV. Is the product of two adjacent sides of the triangle, divided by 2.
   - V. Has only two distinct formulas that use the altitudes and sides of the triangle.
   - VI. More than 50% of the above are true.

3. In *standard* notation:
   - I. The *feet* of the altitudes are denoted by h_a, h_b, and h_c.
   - II. The vertices of the triangle are denoted by lower-case letters.
   - III. The sides of the triangle are denoted by upper-case letters.
   - IV. The altitudes have a special notation that uses the side which is perpendicular to the altitude.