4. History again. Impossible Geometry Problems: Answer Key

RECAP 1: New Vocabulary & Ideas (48 pts)

Check the correct answer. Explain your choice in words and provide details.

1. Euclid's *Elements* are: **(4 pts each)**
   - I. A list of chemical elements found in antiquity.
   - (II.) A book full of geometric axioms and propositions.
   - III. A series of 13 books.
   - IV. A valuable book with antique recipes, translated into numerous languages.
2. *Euclidean constructions* are performed with:
   - I. A ruler and compasses.
   - II. A protractor and a straightedge.
   - (III.) Compasses and a straightedge.
   - IV. A pencil, ruler, and a needle.
3. A *straightedge* is used to:
   - I. Punish children who misbehave.
   - II. Draw a rhombus.
   - III. Measure distances.
   - IV. Find flaws in someone's drawings.
4. With a pair of *compasses* one can:
   - I. Draw ellipses.
   - II. Find his way back to Berkeley.
   - III. Construct a regular 7-gon.
   - (IV.) Aid others in finding the exact midpoint of a segment.
5. A *ruler* was not used by the Greek geometers because:
   - (I.) It is believed that they did not know how to operate with numbers written in the 10-base system.
   - II. It is rumored that they were not smart enough to use the power of the ruler.
   - III. It is known that they did not have computers to help them do arithmetic problems.
   - IV. It is suspected that they liked working only on hard and interesting geometry problems.
6. To find the exact *midpoint* of a segment one needs to:
   - I. Improve his detective abilities.
   - II. Find out how Gauss did it.
   - (III.) Use the properties of a special quadrilateral.
   - IV. Know that the diagonals of a rhombus are not necessarily equal in length.
7. The exact *centroid* of a triangle can be located by:
   - I. Redrawing the triangle on a graph paper and using the properties of the graph paper.
   - (II.) Intersecting the medians of the triangle.
   - III. Finding the approximate place of the center of mass of the triangle using an inflexible model of the triangle and hanging it on a string.
   - IV. Using a GPS device.

8. The address of the Mathematical Sciences Research Institute in Berkeley is "*17 Gauss Way*" because:
   I. The Greeks failed to draw a regular 17-gon.
   II. The street names in Berkeley are funny.
   III. Gauss was 17 years old when he discovered important geometric facts.
   IV. A regular 17-gon is the first in a sequence of polygons drawn by Gauss, using a straightedge and compasses.
9. It is *possible* to:
   I. Draw a rhombus with a ruler and compasses.
   II. Double the volume of the cube with a straightedge and compasses.
   III. Draw a regular heptagon with a straightedge and compasses.
   IV. Divide any angle into three equal sub-angles with a straightedge and compasses.
10. In Berkeley, there is a *street named*:
   I. Euclid.
   II. Archimedes.
   III. Euler.
   IV. Desire.
11. We study *Euclidean constructions* for a number of reasons EXCEPT:
   I. It is a beautiful topic that provides insights to many geometric problems.
   II. It is interesting from a historical perspective to learn about humankind.
   III. We like to make our lives really hard.
   IV. It teaches us algorithms for constructing many shapes.
12. *Euclid*:
   I. Was a famous scholar and scientist in the ancient times.
   II. Wrote one of the most famous books of all times.
   III. Used only a straightedge and compasses in the 465 propositions in his *Elements*.
   IV. All of the above.

RECAP 2: New Theory (15 pts; 3 pts one answer OK + 2 bon. each extra answer)

What new facts did you learn about:

- Rhombi? **(3 pts)**
  - A rhombus is a quadrilateral with four equal sides.
  - The diagonals in a rhombus are perpendicular and bisect each other.
  - The diagonals in a rhombus are not necessarily equal.
- Equilateral triangles? **(3 pts)**
  - A triangle with three equal sides.
  - A triangle with three equal angles.
- Regular polygons? **(3 pts)**
  - A regular polygon is a polygon with equal sides and with equal angles.
  - Not all regular polygons can be drawn with compasses and a straightedge.
  - The Greeks constructed a number of regular polygons.
  - Gauss determined which regular polygons can and which cannot be constructed with compasses and a straightedge.
  - Gauss constructed regular 17-gon, 34-gon, 51-gon, and others.

- Impossible geometry problems? **(3 pts)**
  - There are construction problems which cannot be solved with a straightedge compasses.
  - One such impossible problem is *Circle Squaring*: to draw a square with the same area as a given circle.
  - Another such impossible problem is *Cube Duplication:* to draw a cube with volume twice as big as a given cube.
  - A third such impossible problem is *Angle Trisection:* to divide any angle into three equal angles.
- Drawing tools allowed in Euclidean constructions? **(3 pts)**
  - In Euclidean constructions we can use only a straightedge (without any markings) and compasses.
  - There are good reasons for the Greeks to restrict geometric constructions only to using a straightedge and compasses.

RECAP 3: Algorithms (15 pts)

What is an algorithm? **(3 pts)** Describe the algorithms you learnt so far, using only a straightedge and compasses, to construct:

- An equilateral triangle. **(3 pts)**
- A rhombus. **(3 pts)**
- The midpoint of any segment. **(3 pts)**
- The centroid of any triangle. **(3 pts)**

- An *algorithm* is a sequence of steps that lead to a solution of a problem.
- To draw an *equilateral triangle*:
  1. Draw a segment *AB*.
  2. With centers *A* and *B* draw two circles of radii equal to *AB.*
  3. Where the two circle intersect, mark the points as *C* and *D*.
  4. The triangles *ABC* and *ABD* are both equilateral.
- To draw a *rhombus*,
  1. Follow the same algorithm above for constructing an equilateral triangle.
  2. The quadrilateral *ADBC* is a rhombus.
  3. The two radii in the algorithm do NOT need to be equal to *AB*, but they need to be equal to each other in order to produce a *rhombus*.
- To draw the *midpoint* of a segment *AB*,
  1. Follow the procedure above to construct a rhombus *ADBC* one whose diagonals is *AB.*
  2. Draw the other diagonal *CD* of the rhombus.
  3. The point *M* where the two diagonals meet will be the *midpoint* of *AB.*
- To draw the *centroid* of a triangle *ABC*,
  1. Follow the algorithm above to draw the midpoints of the three sides *AB*, *BC*, and *CA.*
  2. Connect each vertex with the midpoint of the opposite side, in order to draw the three medians.
  3. The point *G* where the three medians intersect will be the *centroid* of the triangle.
  4. It is sufficient to draw and intersect only two of the medians.

RECAP 4: Mathematical Logic
(10 pts + 2 bonus)

- What is the difference between a *theorem* and an *algorithm*? Which needs to be proven?
  - A *theorem* is a general true statement, while an *algorithm* is a sequence of steps to construct something or reach some result. An algorithm could be *part of the proof* of a theorem. **(3 pts)**
  - *Both* a theorem and algorithm need to be proven: that the theorem is true and that the algorithm works and does what it is supposed to do. **(1 pts)**
- Did we *prove* any of the new theory facts, new historical facts, or new algorithms?
  - We did *not* prove them. **(1 pts)**
- How are we so sure they are true?
  - We experimented and checked in specific cases. We relied on facts we had observed before. **(2 pts)**
- What evidence do we have that they are true? **(3 pts for one answer + 2 bonus).**
  - The construction algorithms for an equilateral triangle, a rhombus, a midpoint of a segment, and the centroid all worked out in the specific cases we drew.
  - The diagonals in a rhombus did indeed look perpendicular and bisected each other in the cases we drew.