2. Geometric Constructions: What, Why, and Bits of History

These historical notes are compiled from Wolfram MathWorld, http://www.mathopenref.com/constructions.html, and http://aleph0.clarku.edu/~djoyce/java/elements/elements.html.

Definition 1. In antiquity, geometric constructions of figures and lengths were restricted to the use of only a straightedge and compasses. The Greeks formulated much of what we think of as geometry over 2000 years ago. In particular, the mathematician **Euclid** documented it in his book "Elements". Because of the prominent place Greek geometric constructions held in Euclid's *Elements*, these constructions are known as *Euclidean constructions*.

Question 1. What is Euclid's *Elements*?

Historical Facts. Euclid's *Elements* is one of the most beautiful and influential works of science in the history of humankind. Its beauty lies in its logical development of geometry and other branches of mathematics. It has influenced all branches of science but none so much as mathematics and the exact sciences.

The *Elements* consists of 465 propositions, divided into 13 "books" (an archaic word for "chapters"). The *Elements* were used as a textbook for more than 1000 years in Western Europe, and they went through more than 2000 editions. An Arabic version appears at the end of the 8th century, and the first printed version was produced in 1482. The *Elements* have been studied 24 centuries in many languages starting, of course, in the original Greek, then in Arabic, Latin, and many modern languages.

Definition 2. The word *construction in geometry* has a very specific meaning: the drawing of geometric items such as lines and circles using only compasses and straightedge.

Very importantly, you are not allowed to measure angles with a protractor or measure lengths with a ruler.

Question 2. Which "compasses" do we mean? There is a compass and compasses...

Definition 3. *Compasses* are a drawing instrument used for drawing circles and arcs. It has two legs, one with a point and the other with a pencil or lead. You can adjust the distance between the point and the pencil and that setting will remain until you change it.

This kind of compass has nothing to do with the kind used find the north direction when you are lost. A compass used to find the north direction is usually referred to in the singular - a compass.

The kind we are talking about here is usually referred to in the plural - compasses. This plural reference is similar to the way we talk about scissors - with an 's' on the end.

Question 3. Why do we keep talking about a "straightedge" and not a "ruler"?

Definition 4. A *straightedge* is simply a guide for the pencil when drawing straight lines. In most cases you will use a ruler for this, since it is the most likely to be available, *but you must not use the markings on the ruler during constructions*. If possible, turn the ruler over so you cannot see them.

Question 4. Why did Euclid do it this way, disallowing a protractor and a ruler? Why didn't Euclid just measure things with a ruler and calculate lengths?

For example, one basic construction is *bisecting a segment* (dividing it into two equal parts), i.e., finding its midpoint. Why not just measure the segment with a ruler and divide by 2, just as we did when drawing medians in a triangle?

Historical Conjectures. One theory is that the Greeks could not easily do arithmetic. They had only *whole numbers*, no zero, and no negative numbers. This meant they could not for example divide 5 by 2 and get 2.5, because 2.5 is not a whole number - the only kind they had. Also, their numbers did not use a positional system like ours, with units, tens, hundreds, etc., but more like the Roman numerals. In short, it was quite difficult to do useful *arithmetic*.

So, faced with the problem of finding the midpoint of a line, they could not do the obvious - measure it and divide by 2. They had to have other ways, and this lead to the constructions using compass and straightedge or ruler. It is also why the straightedge has no markings. It is definitely not a graduated ruler, but simply a *pencil guide for making straight lines*. Euclid and the Greeks solved problems graphically, by drawing shapes instead of using arithmetic.

Question 5. OK, even if the Greeks had trouble measuring and doing arithmetic with fractions, WE can! We have rulers and we use decimals for numbers?!

Geometry to the Rescue. Recall that all measures we did to find the midpoints were *approximations*, as long as we weren't using the graph paper! And even on the graph paper, we had to restrict ourselves to placing the vertices of the triangles only on the grid points!

If we want to find the absolute exact midpoint of any segment, we cannot resort to a (regular) ruler! We need geometric tools and a geometric construction to pinpoint the exact midpoint!

In the construction geometry section this summer, following into the footsteps of the ancient Greek geometers, we shall learn how to solve fundamental geometric problems using only compasses and a straightedge. In addition to the centroid of a triangle, we shall construct several other famous points for the triangle.