AMC 8

New Math Concepts, Topics and Problem-Solving Techniques

Custom-Designed

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General Instructions

A. Below are the 8 math areas that appear most often in competitions like Math Kangaroo, AMC 8, Bulgarian Contest, and others:

1. Arithmetic (AR)
2. Algebra (AL)
3. Number Theory (NT)
4. Combinatorics:
   - General Combinatorics (GC) and
   - Combinatorial Geometry (CG)
5. Geometry:
   - Plane Geometry (PG)
   - Spatial Geometry (SG)
6. Optimization (OP)
7. Logic (LO)
8. Probability and Statistics (PR, ST).

Review/study their descriptions, main tasks, objects, and ideas. See if you can connect the problems that you solve with one or more of these areas.

B. Loosely assigning math areas. Many problems can be solved in more than one way, and hence more than one math area can be connected with them. Even one solution may require ideas from different areas of mathematics. It is possible that people will disagree on the math area for certain problems. Thus, view assigning "math areas" to problems as a fun but slightly "imprecise" part of problem-solving that will give you a general feeling and deeper understanding of each problem, especially in how it relates to other problems in mathematics.

8 Most Common Topics in Math Contests

1. Arithmetic

Arithmetic (from Greek ἀριθμός, *arithmos,* “number”) is the oldest and most elementary branch of mathematics, used for tasks from simple day-to-day counting to advanced science and business calculations. It involves the study of quantity, especially as the result of operations that combine numbers. In common usage, it refers to using the four basic operations of *addition, subtraction, multiplication, and division* on small numbers.

2. Algebra

Algebra (from Arabic *al-jebr,* “reunion of broken parts”) comes from the idea that one can perform operations of arithmetic with non-numerical mathematical objects. These objects are *variables* representing either numbers that are not yet known (*unknowns*) or unspecified numbers (*indeterminates* or *parameters*). This allows one to state and prove properties that are true no matter which specific numbers are involved.

3. Number Theory

Number theory is devoted primarily to the study of the *integers* and is often called *“The Queen of Mathematics”* because of its foundational place in mathematics. Number theorists study *prime numbers* and objects made out of *integers* (e.g., fractions like 3/5 called *rational numbers*), and they solve *Diophantine equations*: equations whose solutions are only integers. Number theory is often considered to have given birth to *computer science*.

4. Combinatorics

Combinatorics is the art of counting objects by type, size and other common properties. Combinatorics solves problems from many other areas of mathematics, science, and computer science. In the later 20th century, powerful development made it into an independent branch of mathematics.

5. Geometry

Geometry (Ancient Greek: *geo-* "earth", *-metron* “measurement”) is concerned with questions of *shape, size, relative position of figures, and the properties of space*. Geometry arose independently in a number of early cultures as a body of practical knowledge concerning *lengths, areas, and volumes*. By the 3rd century BC geometry was put into an axiomatic form by *Euclid*. *Archimedes* developed ingenious techniques for calculating areas and volumes. The field of *astronomy* poses lots of geometric problems.

6. Optimization

Optimization is the selection of a best element (for example, the largest or smallest element) from a bunch of things. Optimization is the foundation of *economics* and *ecology*, and it applies to any science and area of human endeavor, from day-to-day life activities such as taking the shortest route to school to global questions of most efficiently supplying food to Earth’s population, or choosing the most efficient orbit for a satellite.

7. Logic

Logic (from the Greek λογική, *logike*) describes the use of valid reasoning in some activity and is featured most prominently in *philosophy*, *mathematics*, and *computer science*. Logic was studied in ancient India, China, Persia, and Greece. In the West, logic was established as a formal discipline by *Aristotle*, who gave it a fundamental place in philosophy.

8. Probability & Statistics

Probability and Statistics are two related but separate disciplines.

Probability theory studies the likelihood than an event will occur, i.e., it answers rigorously the questions "Will a specific event occur?" and "How certain are we that the event will occur?" The level of our "certainty" is called the *probability* of the event to occur. It is a number between 0 and 1:

- 0 means "impossibility": the event never occurs;
- 1 means "certainty": the event always occurs.
- 0.5 means that the event is as likely to occur as not to occur, i.e., it will occur half of the time (0.5=50/100=1/2).
- 0.25, for example, means that the event is not likely to occur, i.e., it will occur less than half of the time; more precisely, it will occur in 1 out of 4 instances (because 0.25=25/100=1/4).
- 0.75, for example, means that the event is likely to occur; it will occur more than half of the time; more precisely, it will occur in 3 out of 4 instances (because 0.75=75/100=3/4).
- The probability can be any number between 0 and 1, including 0 and 1.
- The total probability of all outcomes is always 1. That is, if you add the probability of all outcomes you must always get 1.

Thus, the higher the probability of an event, the more certain we are that the event will occur. The lower the probability of an event, the less certain we are that the event will occur.

A classic example is the toss of a fair coin. The 2 outcomes are "heads" and "tails". Each of them has equal probability (or equal chance) to occur because the coin is "fair". Since the two probabilities add up to 1 and they are equal, each probability is 1/2. We can say equivalently that there is a 50% chance of getting either "heads" or "tails".

Note that *percentages* (%) measure how many *hundredths* there are. For example, 1% (one percent) means 1 hundredth: 1/100=0.01, and 37% means 37 hundredths: 37/100=0.37. One can convert probabilities from decimal numbers to fractions to percentages. For example, some common probabilities are:

- 0.5=50/100=50%=1/2 (occurs 1 in 2 instances);
- 0.25=25/100=25%=1/4 (occurs 1 in 4 instances);
- 0.33=33/100=33%≈1/3 (occurs≈1 in 3 instances);
- 0.75=75/100=75%=3/4 (occurs 3 in 4 instances);
- 0=0/100=0% (occurs 0 in all instances; i.e., never occurs);
- 1=100/100=100% (occurs 1 in 1 instances, or 100 in 100 instances; i.e., always occurs).

How to calculate the probability of a certain event?

- For example, if we roll a fair dice, what is the probability that the number we get is divisible by 3?

1. Calculate the number of all possible outcomes. In our example, all outcomes of the dice are 1, 2, 3, 4, 5, and 6, and each is equally probable; i.e., the probability getting 4 is 1/6 and the probability of 6 is also 1/6. Thus, the total number of possible outcomes is 6.
2. Calculate the number of desirable outcomes that correspond to the event you are interested in; in our example, we want to get a number divisible by 3, i.e., 3 or 6. These are 2 desirable outcomes.
3. Divide the number of desirable outcomes by the number of total outcomes:

Probability of our event = ———————— = – = – = 0.333333…≈ 33%

4. Conclude that the probability of the event is 1/3 or about 33%, i.e., the event will occur on the average 1 out 3 times. That is, if you roll the dice 21 times, most likely in 7 of these rolls you will get a number divisible by 3 (i.e., you will roll 3 or 6) and in the remaining 14 rolls you will get a number not divisible by 3 (i.e., you will roll 1,2,4, or 5).

Statistics is the study of the collection, organization, analysis, interpretation and presentation of data. Its mathematical foundations were laid in the 17th century with the development of probability theory by *Blaise Pascal* and *Pierre de Fermat*. Mathematical probability theory arose from the study of games of chance, although the concept of probability was already examined in *medieval law* and by *philosophers* such as *Juan Caramuel*. Today, statistics is widely employed in *government, business, and natural and social sciences*.

A main task in statistics is to calculate various **averages** for given numbers (or data). The most common average is the so-called

Arithmetic mean (or just **average**) = ——— = ——— .

Median = the **middle** of all numbers when arranged in increasing order:

- When *is odd* there will be a unique middle number. For example, if we have 5 numbers total, then the 3rd number will be the middle number and hence it will be the median: gives .
- When *is even* there will be two "middle numbers". For example, if we have 6 numbers total, then the 3rd and the 4th number will compete to be the middle numbers: a makes and the two middle numbers. The median will be the average of and : ——— .

Mode = The value that appears most often. It may or may not be unique. For example, in the set {1,2,2,5,6,7,8,8,8,10}, the mode is 8; but in the set {1,2,2,5,5,5,6,7,8,8,8,10} there are two modes: 5 and 8.

For short, we will denote Probability by **PR** and Statistics by **ST**.

Geometry

A major task in Geometry is to find the perimeters and areas of figures, and the volumes of solids. Fill in the following table with the correct formulas for perimeters and areas of quadrilaterals and triangles.

Name of figure	**Drawing**	**Perimeter**	**Area**
1. Square			
2. Rectangle			
3. Parallelogram			
4. Rhombus			
5. Trapezoid			
6. Right triangle			
7. Any triangle			
8. General Quadrilateral			

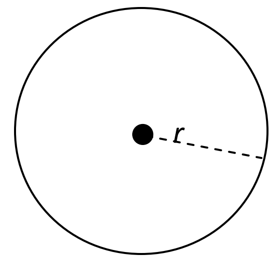
Circles present a special challenge in finding their perimeters and areas, as the formulas involve the "mysterious" number π:

Perimeter $P = 2\pi r$

$\pi = 3.141592654\ldots \approx 3.14 \approx$ —

Theorem. For a *circle* of radius *r*:

a) Its perimeter is equal to $2\pi r$.
b) Its area is equal to πr^2.

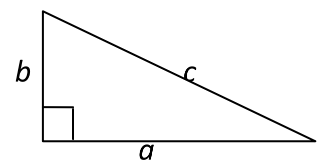


Area $A = \pi r^2$

Right triangles play an important role in geometry. In fact, one of the most famous theorems in mathematics is about the lengths of sides in a right triangle. A right triangle has two *legs* (the sides that make its right angles) and a *hypotenuse* (the side that lies across the right angle).

Pythagorean Theorem. In a right triangle with legs *a* and *b* and hypotenuse *c*:

$$a^2 + b^2 = c^2.$$



For example, in a right isosceles triangle with legs $a = b = 1$ the hypotenuse *c* must satisfy:

$$1^2 + 1^2 = c^2, \text{ i.e., } c^2 = 2.$$

The operation that we use to calculate *c* is called "taking the *square root*":

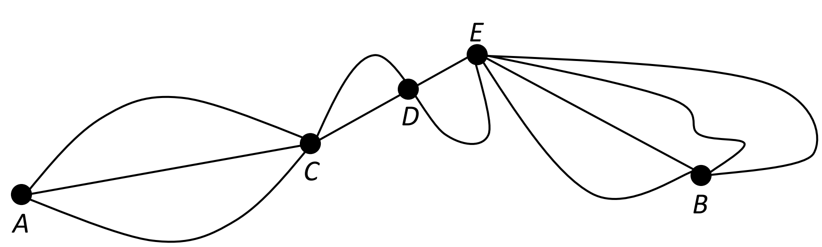
$$c = \quad = 1.414213562\ldots.$$

Indeed, if you square the result, you will get 2:

$$(1.414213562\ldots)^2 = 2.$$

Combinatorics

The major job of Combinatorics is to count anything; for example, to count the number of ways to get from place *A* to place *B*. If there are intermediate places *C, D,* and *E* between *A* and *B*, then we need to *multiply* all numbers of options along the way:



ways to get from *A* to *B* = (# ways from *A* to *C*) x (# ways from *C* to *D*) x

x (# ways from *D* to *E*) x (# ways from *E* to *B*) =

= 3 x 2 x 2 x 4 = 48 ways.

Main Objects of study in each area

Arithmetic

Main object: *expressions with numbers* and the four basic operations {+, -, ∗, ÷}, e.g., $6 + (3 * 5 - 7) \div 4$

Algebra

Main object 1: *expressions with variables* and the four basic operations {+, -, ∗, ÷}, e.g., $6 + (x*5-7) \div 4$

Main object 2: *equations with variables* and four basic operations {+, -, ∗, ÷}, e.g., $6 + (x*5-7) \div 4 = 12$

Main object 3: *systems of equations with variables* and four basic operations {+, -, ∗, ÷}, e.g., $x + y = 12$, and $x = 3y + 4$. (Same number of equations and variables!)

Number Theory

Main object 1 (Divisibility/Remainders): Use divisibility and remainders to find on which day of the week (which month or year, etc.,) an event falls.

Main object 2 (Integers): Study properties of integers, such as their possible sums, differences, products, their digits, digits being different or the same, being divisible by something, how many divisors, being prime, being consecutive, etc., and find which integers or how many integers have these properties.

Main object 3 (Sequences, Patterns): Construct, study, and use sequences.

Main object 4 (Diophantine equation): Solve an equation with 2 variables, looking only for integer answers. Solve two equations in 3 variables, looking only for integer solutions. Etc.

Combinatorics

Main object 1 (General Combinatorics): numbers, objects, anything that can be counted with integers;

Main object 2 (Combinatorial Geometry): geometric objects in the plane, in space, anywhere.

Geometry

- Main object 1 (Plane Geometry): geometric shapes in the plane, e.g., points, lines, segments, rays, polygons, circles, ellipses, etc.
- Main object 2 (Spatial Geometry): geometric solids in space, e.g., polyhedrons (cubes, parallelepipeds, and prisms), solids with some round edges (cones, cylinders), and spheres.

Optimization

- Main object 1 (Rounding Numbers): rounding numbers up or down to find an optimal answer.
- Main object 2 (Game Theory): study games, (e.g., wins, losses, ties, and number of such).
- Main object 3 (Graph Theory): study properties of graphs: structures made of points (vertices) some of which are connected by segments (edges), e.g., like a map of a country where the vertices are the towns and an edge between two vertices shows that there is a road between these towns.

Logic

- Main object 1 (General Relations): Establish the proper order/ relations between objects, using given data and logic.
- Main object 2 (Time Problems): Place objects or events in correct time (chronological) order, using the given data and logic. Calculate number of events/objects sharing same time features.

Probability & Statistics

- Main object 1 (Probability): Find the probability that an event occurs.
- Main object 2 (Statistics): Calculate various averages of numbers, e.g., the arithmetic mean (or called just the average), the median, the mode, etc.
- Main object 3 (Statistics): To approximate various quantities.

Worksheet 1 (Use AMC 8, 2011)

AR, AL, PG, SG, CG, GC, NT, OP, LO, PR, or ST?

Prob. & Answer	Area(s)	List one solution: how did you solve the problem? What was the main idea that helped you solve the problem?
#1 Answer:		*Hint:* "Change" indicates *subtraction*. What to subtract and from where?
#2 Answer:		*Hint:* Brute-force calculation is the cure here. ☺
#3 Answer:		*Hint:* Use graph paper. Count brute-force first; then try in a sleek way.
#4 Answer:		*Hint:* Read about mode, median, and mean in the Statistics section.
#5 Answer:		*Hint:* Convert correctly from minutes to hours, and to days.

Worksheet 2 (Use AMC 8, 2011)

AR, AL, PG, SG, CG, GC, NT, OP, LO, PR, or ST?

Prob. & Answer	Area(s)	List one solution: how did you solve the problem? What was the main idea that helped you solve the problem?
#6* Answer:		*Hint:* Draw a picture. If you know Venn diagrams, draw such a diagram.
#7 Answer:		*Note*: Is there a super fast way to move all important stuff in one place?
#8 Answer:		*Hint:* Make a picture. List all possibilities. Think of a neat way to do this.
#10 Answer:		*Hint:* Try out brute force first, adding up more money and more mileage.
#13* Answer:		*Hint:* This problem should somehow remind you of #6! *Percents* mean "how many hundreds," e.g., 20% means 20 hundreds, or 20/100=1/5.

Worksheet 3 (Use AMC 8, 2011)

AR, AL, PG, SG, CG, GC, NT, OP, LO, PR, or ST?

Prob. & Answer	Area(s)	List one solution: how did you solve the problem? What was the main idea that helped you solve the problem?
#9* Answer:		*Hint:* This is a graph of a function. How do we read it? Distance = time x speed. What does *speed* equal to? *Average* speed refers to the whole trip.
#11 Answer:		*Hint:* This could be done by brute-force calculations, or think: "By how much does every day Sasha study more or less than Asha?"
#12* Answer:		*Hint:* Draw the table, sit somewhere Angie, and see in how many ways you can then seat the other people. Read about *probability* earlier.
#14 Answer:		*Hint:* How many girls are in the first school? What fraction of the students there are girls? Draw a picture. Repeat for the second school.
#15 Answer:		*Hint:* The number 5 loves most to be multiplied by which other number? Can you arrange for this to happen?

Worksheet 4 (Use AMC 8, 2011)

AR, AL, PG, SG, CG, GC, NT, OP, LO, PR, or ST?

Prob. & Answer	Area(s)	List one solution: how did you solve the problem? What was the main idea that helped you solve the problem?
#17 Answer:		*Hint:* Split 588, little by little, into a product of *prime* numbers.
#18* Answer:		*Hint:* List all possibilities for the first two rolls. Which are "good"?
#21 Answer:		*Hint:* Knock down possibilities, but be careful not to eliminate too much.

#22** Answer:		*Hint:* Concentrate only on the last digits only. Discard all other digits. Keep multiplying 7 by itself. Do you see a pattern in the last digits?
#23* Answer:		*Hint:* Draw 4 slots for the digits. Which numbers are divisible by 5? Split the counting into two cases. How many are options there for each slot?
#24* Answer:		*Hint:* Think of odd and even numbers. Which primes are even?

Worksheet 5 (Use AMC 8, 2011)

AR, AL, PG, SG, CG, GC, NT, OP, LO, PR, or ST?

Prob. & Answer	Area(s)	List one solution: how did you solve the problem? What was the main idea that helped you solve the problem?
#16** Answer:		*Hint:* Need to know the area of a triangle and the Pythagorean Theorem.
#19* Answer:		*Hint:* Redraw on graph paper. Count the rectangles containing a specific side, and then erase that side. Choose sides from the outside first.
#20** Answer:		*Hint:* Draw altitudes from *A* and from *B,* to split the trapezoid into simpler shapes. Will need the Pythagorean Theorem for the triangles.
#25** Answer:		*Hint:* Fold the shaded parts inside to cover the white square. How much more is the area of the large square than the area of the white square?