Berkeley Math Circle

February 17, 2015

1. Find the least $n \in \mathbb{N}$ such that among any n rays in space sharing a common origin there exist two which form an acute angle. What about n rays in d-space (vectors in $\mathbb{R}^d$?

2. Prove that for any integers a,b, the equation $2abx^4 - a^2x^2 - b^2 - 1 = 0$ has no integer solutions.

3. Take three points inside a rectangle. They determine a triangle and we take a fourth point inside this triangle. Prove that at least one of the three concave quadrilaterals formed by these four points has the perimeter smaller than the perimeter of the rectangle.

4. Consider the sequence $(3^{2^n} + 1)_{n \geq 1}$.

   i) Prove that there exist infinitely many primes such that each divides some term of the sequence.

   ii) Prove that there exist infinitely many primes that do not divide any of the terms of the sequence.

5. Consider the sequence $(a^n + 1)_{n \geq 1}$, with $a > 1$ a fixed integer.

   i) Prove there exist infinitely many primes, each dividing some term of the sequence.
   ii) Prove there exist infinitely many primes, none dividing any term of the sequence.

6. A set S of unit cells of an $n \times n$ array, $n \geq 2$, is called **full** if each row and each column of the array contains at least one element of S and if we remove any of the elements, the property no longer holds. A full set having maximum cardinality is called **fat**, while a full set of minimum cardinality is called **meagre**.

   i) Determine the cardinality $m(n)$ of the meagre sets. How many of them there are?

   ii) Determine the cardinality $M(n)$ of the fat sets. How many of them there are?

7. If for some positive integer n, $2^n - 1$ is prime, then so is n.

8. For a positive integer a, define a sequence of integers $x_1, x_2, \ldots$ by letting $x_1 = a$ and $x_{n+1} = 2x_n + 1$ for $n \geq 1$. Let $y_n = 2^{x_n} - 1$. Determine the largest possible k such that, for some positive integer a, the numbers $y_1, \ldots, y_k$ are all prime.