Dissections, Cavalieri's Theorem, and the Sweeping Tangent Theorem

Austin Shapiro • Berkeley Math Circle (Adv) • January 20, 2015

No integrals allowed!

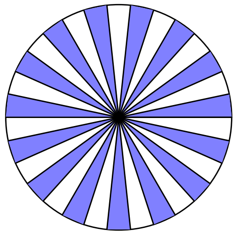

Figure 1

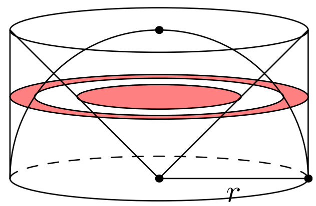

Figure 2

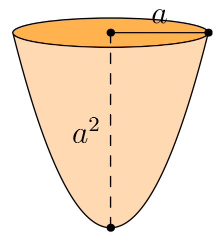

Figure 3

No integrals allowed!

1. *(Archimedes)* By rearranging the thin circular sectors in Figure 1, derive the formula for the area of a circle of radius r.
2. *(AHSME, 1987)* A cube of cheese $C = \{(x, y, z) : 0 \le x, y, z \le 1\}$ is cut along the planes $x = y$, $y = z$, and $z = x$. How many pieces are there? (No cheese is moved until all three cuts are made.)
3. Show that the volume of a tetrahedron is $A \cdot h/3$, where A is the area of one face and h is the altitude from the fourth vertex to the plane of that face. Hence, show that a similar formula holds for a pyramid or cone over any plane region.
4. *(Archimedes)* A hemisphere and a right circular cone are inscribed in a cylinder of height and radius r, as shown in Figure 2. A cross-section of all three solids is taken parallel to the base of the cylinder. How do the cross-sectional areas of the three solids relate? Use your result to derive the formula for the volume of a sphere.
5. Hence, what is the surface area of a sphere of radius r?
6. Let a be a constant. The area lying below the line $y = a^2$ and above the parabola $y = x^2$ is revolved around the y-axis to sweep out a paraboloidal solid (Figure 3). Determine the volume of the solid.

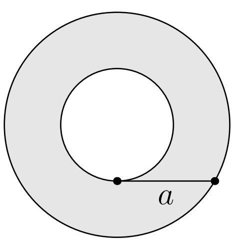

Figure 4

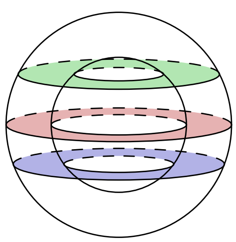

Figure 5

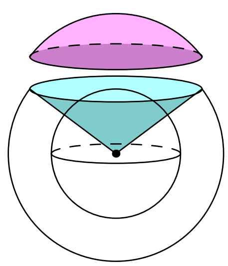

Figure 6

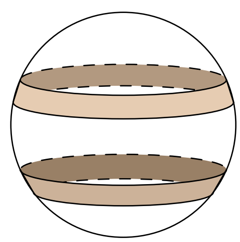

Figure 7

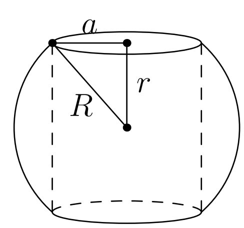

Figure 8

7. Determine the area of the annulus in Figure 4.
8. A solid shell bounded by two concentric spheres of radii R and r (where $R > r$) is sliced by planes. Show that the cross-sections intersecting both spheres all have the same area (Figure 5).
9. Continuing Problem 8, find the volume of the spherical cap cut off the larger sphere by a plane tangent to the smaller sphere (Figure 6).
10. Continuing, find the volume of the cone whose vertex is at the common center of both spheres and whose base is the base of the spherical cap from the last problem (Figure 6).
11. Hence, determine the surface area of the spherical cap (only the curved surface). The ratio of this area to R^3 is known as the *solid angle* subtended by the cap; the units are *steradians*.
12. A sphere of radius R is cut by two parallel planes (Figure 7). Show that the surface area of the band between the planes depends only on the distance between the planes.
13. A hole of radius a is drilled through a sphere of radius R (Figure 8). Find the volume of the remaining "napkin ring".

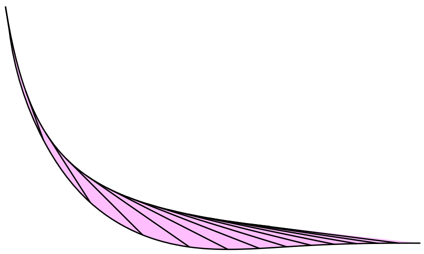

Figure 9

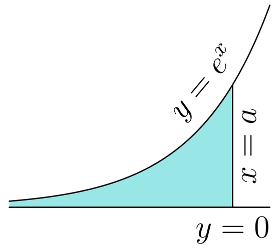

Figure 10

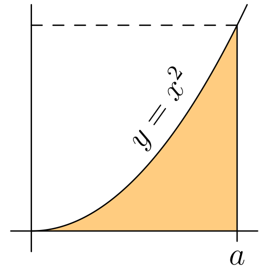

Figure 11

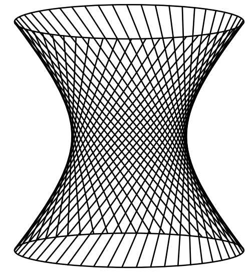

Figure 12

14. *(Mamikon)* A southbound bicyclist turns left onto an eastbound street. If the length of the bicycle (the distance between the two wheels' points of contact with the pavement) is a, what is the area bounded between the tracks made by the two tires? (See Figure 9. You can draw your own region of this kind with the side of a piece of chalk.)

14½. A child walks to the right along the x-axis, dragging a toy duck on a string of length 1. The child begins at the origin, and the toy begins at $(0, 1)$. What is the area of the region bounded on the left by the y-axis, above by the toy's path, and below by the x-axis? (The toy's path is a classical curve called a *tractrix*.)

15. The function $f(x) = e^x$ is the unique function (up to constant scaling) which grows at a rate equal to itself. That is, the line tangent to the graph of $y = e^x$ at the point (t, e^t) has slope e^t. Use this property to find the area bounded above by the graph of $y = e^x$, below by the x-axis, and on the right by the line $x = a$ (Fig. 10).

16. Determine the area in the first quadrant lying below the parabola $y = x^2$ and to the left of the line $x = a$ (Figure 11). This can be done with Cavalieri's Theorem or with Mamikon's Sweeping Tangent Theorem. Archimedes had a third method—he tiled the space above the parabola with triangles.

    For the sweeping tangent approach, note that the line tangent to the parabola $y = x^2$ at the point (t, t^2) has slope $2t$.

17. Generalize Problem 16 to the graph of any power function, $y = x^p$.

18. When line ℓ_1 is rotated around another line ℓ_2 skew but not perpendicular to ℓ_1, a surface called a *hyperboloid of one sheet* is swept out. Figure 12 shows such a surface, symmetrically truncated by planes perpendicular to ℓ_2. Find the volume enclosed by the hyperboloid between those two planes, in terms of the radius of the central cross-section, the radius of the base, and the height.

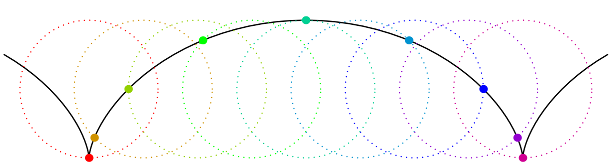

Figure 13

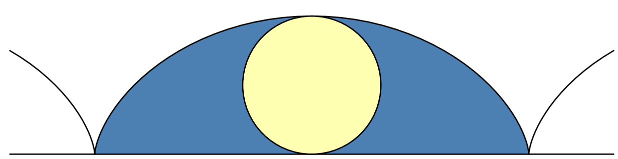

Figure 14

19. A *cycloid* is the path traced by a point on the circumference of a wheel as the wheel rolls along a line without skidding (Figure 13). Determine the ratio of the area of one "arch" of the cycloid to the area of the wheel (Figure 14).

20. Two cylindrical pipes of radius r meet head-on at right angles. What is the volume of their junction?

21. Let $R > r > 0$. A circular disc of radius r, centered at $(R, 0)$ in the xy-plane, is revolved around the y-axis to sweep out a *solid torus*. Determine the volume of this solid.

22. *(IMO shortlist, 1988)* In tetrahedron $ABCD$, let K, L be the midpoints of edges AB, CD respectively. Prove that every plane that contains the line KL divides the tetrahedron into two parts of equal volume.

23. Let Γ be a convex, closed plane curve. Let R_ϵ be the plane region consisting of all points inside Γ or within ϵ units of Γ. How can we find the area of R_ϵ?

Further reading: T. Apostol and M. Mnatsakanian, *New Horizons in Geometry.*