MATHEMATICS OF PEG SOLITAIRE

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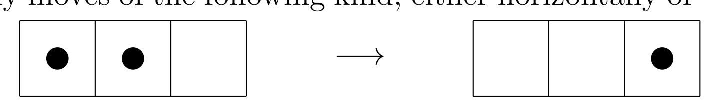
This BMC is based on [BCG]. The peg solitaire game is typically played on a board with the following shape:

		a	b	c		
		d	e	f		
g	h	i	j	k	l	m
n	o	p	x	P	O	N
M	L	K	J	I	H	G
		F	E	D		
		C	B	A		

We labelled the places in order to make it easy to describe moves. Each square in the picture represents a hole in the actual game, and the holes may be empty or filled by a peg. The aim is to go from some initial position (the standard one is: pegs in all positions except the central one), to a final one (typically: just one peg left on the board), using only moves of the following kind, either horizontally or vertically:

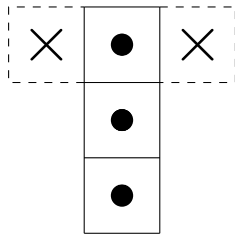


One can say that one peg jumps over another one which is eaten.

We will explore several mathematical aspects of this game. But first of all, can you solve it? Before reading the sequel, try the game a few times!

1. Tools to solve peg solitaire

1.1. **The 3-peg purge.** Assume one of the spaces marked by a cross is occupied, while the other one is empty. Can you see how to get rid of the three pegs in line, while the position of the rest of the pegs is unchanged at the end of the process?

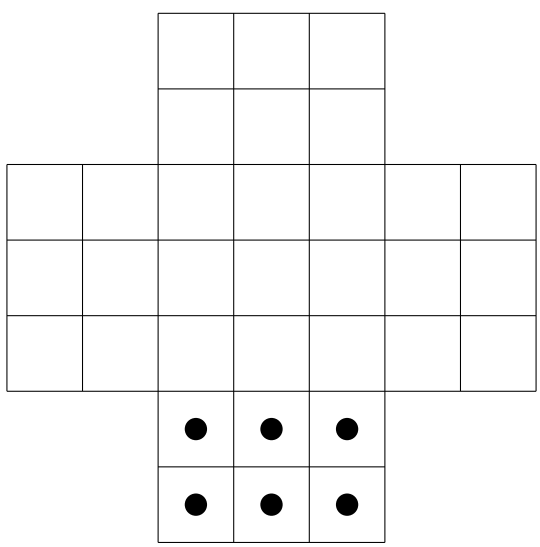


Since the additional peg participates to the process but remains unchanged at the end, one can call it a catalyst.

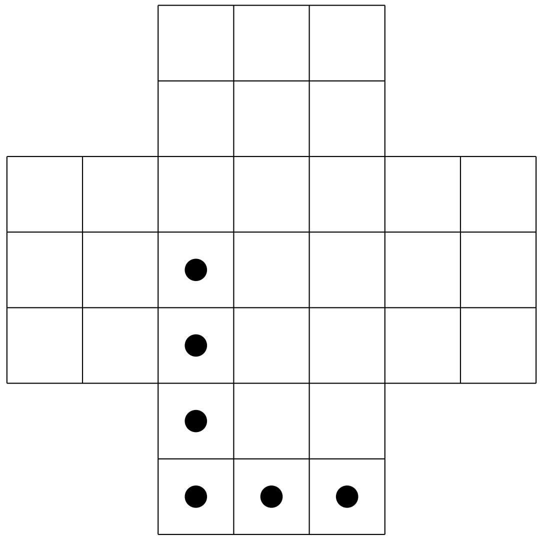
1.2. **The 6-peg purge.** The 3-peg purge is neat, since it allows you to get rid of 3 pegs without modifying the position otherwise. Using this kind of sequence of moves makes it easy to plan many moves in advance in your head (maybe even from the beginning to the end!).

On the other hand, it requires space on both sides, so we cannot use it on the boundaries. It suggests, though, that groups of 3 pegs in a row can possibly be replaced by nothing.

Using a catalyst, can you find a way to get rid of the following 2×3 rectangle in the boundary? (You have to figure out where to place the crosses.)

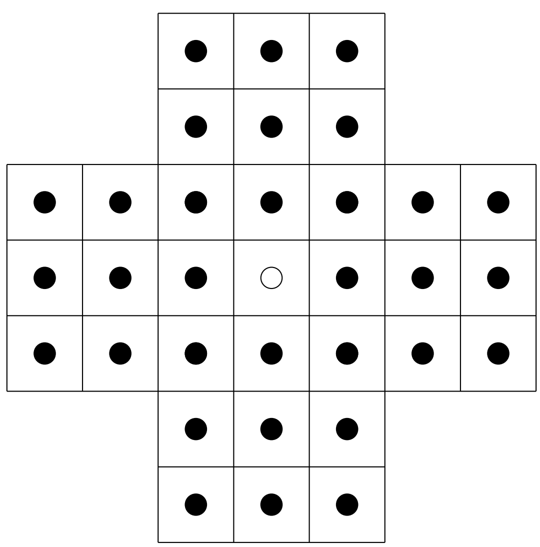


1.3. **The L-purge.** Same question for this L-shape.



2. Solving the game

2.1. **The standard position.** Using the tools above, solve the game in the standard position, with an empty spot in the middle of the board.



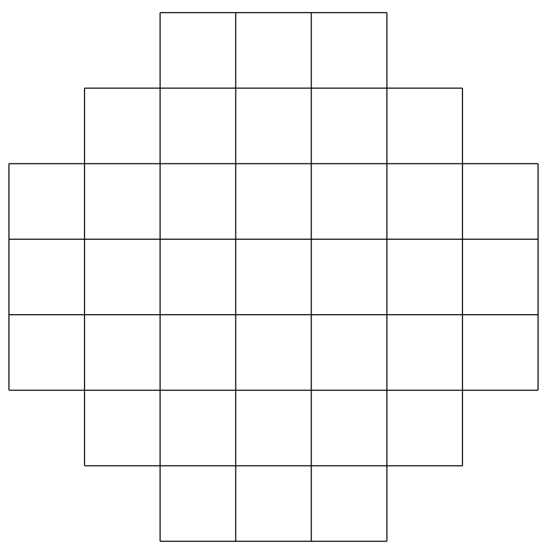
2.2. **Other positions.** You can try to do the same for all initial positions with just one missing peg. Taking symmetries into account, how many cases do you have to consider?

Can you predict where the last peg will be? Can you prove it?

Variant: choose a particular peg and mark it somehow to distinguish it from the other ones. Can you solve the game so that this peg is the last survivor?

If you have done all this, you can try positions where two pegs are missing (then you may want to finish with two pegs at the initially empty places)...

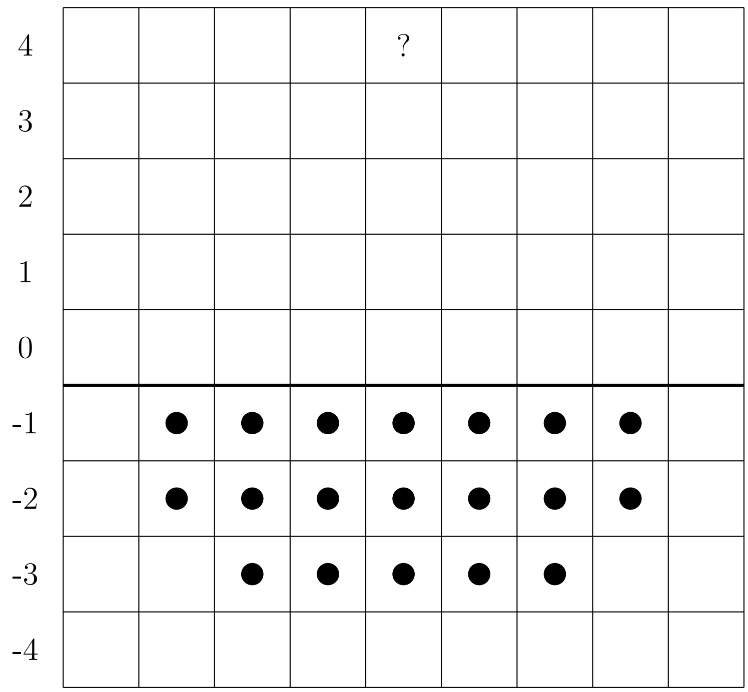
2.3. **Another board shape.** One can also find the following board shape:



Try to solve it in several positions (with one peg removed). Tell me about your progress.

3. The army

A number of solitaire men stand initially on on side of a straight line beyond which is an infinite empty desert. How many men do we need to send a scout just 0, 1, 2, 3, 4, 5, . . . , paces into the desert?



4. General scout problem

A peg solitaire position in the infinite plane can be described by a part of $\mathbb{Z}^2 = \{(x, y) \mid x, y \in \mathbb{Z}\}$, whose elements are the coordinates of the pegs.

Define a distance on $\mathbb{Z}^2$. Define the distance of a point of $\mathbb{Z}^2$ to a part of $\mathbb{Z}^2$. The general scout problem is: given a (nonempty) initial position X (a part of $\mathbb{Z}^2$), how far can you send a scout from X? Is there a universal bound for the maximal distance that can be achieved, independently of X?

Appendix A. The quadratic equation

Consider the quadratic equation $ax^2 + bx + c = 0$, where a, b, c are numbers (say real numbers), and $a \neq 0$. The discrimant of this equation is $\Delta := b^2 - 4ac$.

(1) If $\Delta > 0$, then there are two real solutions:

$$\frac{-b \pm \sqrt{\Delta}}{2a};$$

(2) If $\Delta = 0$, then there is only one ('double') real solution: $-\frac{b}{a}$;
(3) If $\Delta < 0$, then there is no real solution.

Appendix B. Geometric series

If $|q| < 1$, then the series (infinite sum) $\sum_{i=0}^{\infty} q^n = 1 + q + q^2 + \cdots$ converges, and

$$\sum_{i=0}^{\infty} q^n = \frac{1}{1-q}.$$

References

[BCG] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. Winning Ways for your Mathematical Plays, vol. 2. Academic Press, 1982.