Minkowski addition and Minkowski decomposition
of convex lattice polygons in the plane – and mixed volume

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1 Polygon addition

Problem 1.1 (Robot translation) *Given a set of obstacles in the plane and a robot that can move among them by translation, describe the space of allowed positions, or configurations, as a set of points.*

Idea (Configuration space): Shrink the robot to a point, grow the obstacles accordingly. Inflated obstacles constitute occupied space. The rest is free *configuration space:* each point is an allowed robot position.

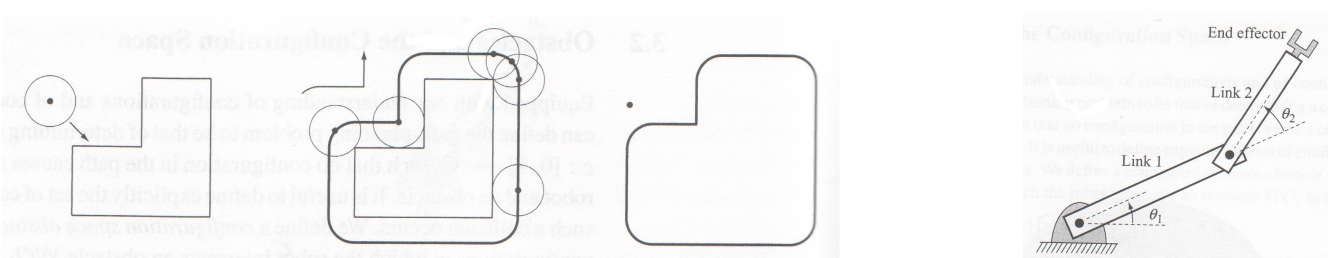

Figure 1: (a) Disk-robot and rectangular obstacle. (b) Robot with two rotational degrees of freedom.

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In Figure 1(a), the disk-robot is represented by its center. We slide it around the obstacle and mark the trajectory of its center. Equivalently, we add the disk to the obstacle by Minkowski addition, see below.

Problem 1.2 (Robot rotation) *Consider the robot arm in Figure 1(b), comprised of 2 straight-line segments. What is the set of allowed configurations of its end-effector in the space of angles θ_1, θ_2?*

Let us restrict attention to polygonal obstacles, or polygons, which are *convex*, hence we exclude the obstacle of Figure 1(a). *Polygons* in the plane are sequences of straight-line segments, called edges, where every two consecutive edges share a point, called vertex. Our polygons lie in the real plane, denoted by $\mathbb{R}^2$. Any point in the plane defines a vector rooted at the origin. For vectors or points a, b in the plane, let $\langle a, b\rangle \in \mathbb{R}$ denote the inner product of the corresponding vectors. For a polygon P and vector v, let $h(P, v) = \sup\{\langle p, v\rangle : p \in P\} \in \mathbb{R}$ be the extreme inner product with v over P; the maximum is actually attained in P. The set $\{p \in P : \langle p, v\rangle = h(P, v)\}$ is either a vertex or an edge of P. Vector v is an outer normal of this vertex or edge. Conversely, if we are given vectors $v_1, \ldots, v_n \in \mathbb{R}^2$ that positively span the plane, and values $h_1, \ldots, h_n \in \mathbb{R}$, a polygon is obtained as the intersection of the corresponding half-planes: $P = \{p \in \mathbb{R}^2 : \langle p, v_i\rangle \leq h_i,\ i = 1, \ldots, n\}$.

For polygons in $\mathbb{R}^2$ there is a natural associative and commutative operation which generalizes vector addition and is called Minkowski addition.

Definition 1.3 *For any two polygons $P, Q \subset \mathbb{R}^2$, their* Minkowski sum *is*

$$P + Q = \{p + q \mid p \in P, q \in Q\}.$$

We call P and Q the summands *of $P + Q$.*

Problem 1.4 (Kinematic definition) *Prove that $P+Q$ is the set covered by translating Q by all points $p \in P$: $P + Q = \bigcup_{p \in P}(p + Q)$.*

Several examples are shown in Figure 2. The definition extends to arbitrary sets in $\mathbb{R}^2$ and, in fact, to any dimension: the middle Figure 1(a) shows the sum of a disk with a non-convex polygon.

Problem 1.5 *If P, Q are convex, then $P + Q$ is convex.*

Problem 1.6 *The face of $P+Q$ with outer normal $v \in \mathbb{R}^2$ is the sum of those faces of P and Q that have outer normal v.*

It now follows that

Corollary 1.7 *The vertices of $P+Q$ are sums of vertices from P and Q. Every edge of $P+Q$ is determined uniquely as the Minkowski sum of an edge and a vertex from P and from Q, respectively, or the sum of two strongly parallel edges (same outer normal) from P and Q.*

Problem 1.8 *Describe zonotopes, which are Minkowski sums of k non-parallel edges, hence parallelograms are zonotopes for $k = 2$. Can you always partition the zonotope into parallelograms? How many?*

Problem 1.9 *What are the faces of the Minkowski sum of two 3-dimensional polytopes?*

Given a finite set of points, their *convex hull* is the smallest (convex) polygon that includes them all. The Minkowski sums of convex polygons can be computed as the convex hull of all sums $a + b$ of *vertices* of P and Q respectively. If each summand has n vertices, this implies computing n^2 vertex sums and, then, constructing their convex hull, which requires about $n^2 \log n$ operations.

For a more efficient method, let us recall Corollary 1.7. It follows that the set of outer normals to edges of $P + Q$ is the union of the sets of the outer normals to edges of P and Q. The method starts at vertex $p \in P$ and vertex $q \in Q$, and shall walk around each polygon in, say, counter-clockwise order. At the same time, we shall be defining $P + Q$ as a sequence of edges, their endpoints being the sum's vertices, starting from $a + b$. Let e be the edge of P or Q with smallest slope, at our current position. Then e becomes the next edge in $P + Q$, and we move to the endpoint of e on its polygon.

Formally, this algorithm sorts the n outer normals of P and of Q, then merges the two lists, thus obtaining the sorted outer normals of $P + Q$. This algorithm takes a linear number of operations in n. Therefore, it is clearly faster than computing the convex hull and, also, optimal in its dependence on n.

2 Polygon decomposition

We study the following "inverse" operation of Minkowski addition, for *lattice* polygons, i.e., having vertices with *integer coordinates.*

Definition 2.1 *Given a lattice polygon S, the* MINKOWSKI-DECOMPOSITION *problem is to decide whether there exist lattice polygons P, Q such that $P+Q = S$, where P, Q are not points; equivalently, neither equals S. If so, find P.*

If $S = P + Q$ then, having determined P, it is easy to compute Q. So it suffices to find one summand, if it exists. Excluding point summands implies that we also exclude S from being considered a summand of itself. In other words, we are only interested in nontrivial (or proper) summands; see Figure 2 for examples.

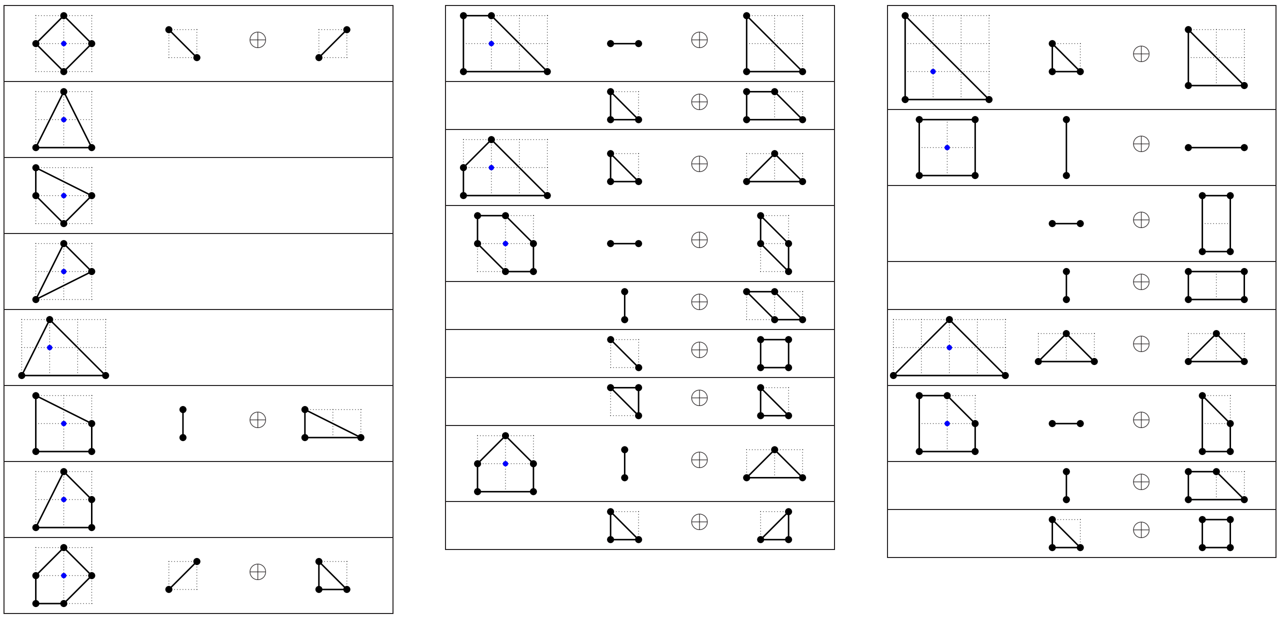

Figure 2: Minkowski decompositions of all 16 lattice polygons with one interior lattice point.

Let us, for a moment, suppose that we are looking for decompositions where P, Q share no parallel edges. Then, using Corollary 1.7, summand P is the polygon defined by a subset of the edges in S whose vector sum is zero. However, since edges in S may be the sum of two strongly parallel edges from P, Q, we should be more careful. Let lattice polygon S be defined by vertices $v_0, v_1, \ldots, v_{n-1} \in \mathbb{Z}^2$. Every edge is represented by an integer vector, namely $u_1 = (v_1 - v_0), \ldots, u_n = (v_0 - v_{n-1}) \in \mathbb{Z}^2$. We write each integer vector u_i as the product of a "primitive" integer vector e_i and an "integer length" d_i:

$$u_i = d_i e_i, \;\; d_i \in \mathbb{N}^*, \, e_i \in \mathbb{Z}^2, \;\; i = 1, \ldots, n,$$

where the absolute values of the coordinates of e_i are coprime. A lattice polygon is a summand of S iff its edge set equals

$$\{k_i e_i, \, i = 1, \ldots, n\} \;\; \text{for some } k_i \in \mathbb{N}, \;\; 0 \le k_i \le d_i, \;\; \text{s.t. } \sum_{i=1}^{n} k_i e_i = (0, 0). \tag{1}$$

In other words, the summand is defined by subvectors of the edge vectors of S, whose vector sum is zero. If $k_1 = \cdots = k_n = 0$ then we obtain a trivial point summand. If $k_i = d_i$, $i = 1, \ldots, n$, then we trivially obtain S. For the summand to be nontrivial, at least one k_i must be positive and at least one k_i must be $\le d_i - 1$.

One algorithm is to reduce MINKOWSKI-DECOMPOSITION to computing a subset of integers that sum up to zero. This corresponds to the following hard problem[1].

[1] It is a famous NP-complete problem.

Definition 2.2 *Given a set (or multiset) of n integers,* SUBSET-SUM *is the problem of deciding whether there exists a non-empty subset, such that its elements add up to zero.*

An instance of MINKOWSKI-DECOMPOSITION can be transformed to an instance of SUBSET-SUM, such that the former admits a solution iff the latter does. Moreover, the solution sets are in bijective correspondence. Given a lattice polygon S with primitive edge vectors $e_i = (e_{ix}, e_{iy}) \in \mathbb{Z}^2$, $i = 1, \dots, n$, let

$$E = \max_{i=1,\dots,n} \max\{|e_{ix}|, |e_{iy}|\}, \quad D = \max_{i=1,\dots,n} d_i.$$

The input to SUBSET-SUM shall be d_i copies of integer a_i, where

$$a_i = e_{ix} + nDE \cdot e_{iy} \in \mathbb{Z}, \quad i = 1, \dots, n,$$

except for a_1 which is given in $d_1 - 1$ copies.

Problem 2.3 *Prove that S admits a nontrivial decomposition, where one summand has edges $k_i e_i$ as in equation (1), iff there is a non-empty subset of all a_i's that sums up to zero, which includes a_i a number of k_i times, with $k_1 < d_1$ and $k_i \leq d_i$, $i = 2, \dots, n$.*

The constraint $k_1 \leq d_1 - 1$ does not reduce generality since it must hold for at least one of the two nontrivial summands, if they exist. Subset sum is solved with a number of operations which is proportional to $n^3 D^3 E^2$.

Problem 2.4 *Let $v_1, \dots, v_n$ be the outer normals to the edges of S, and $h(S, v_i) = \sup\{\langle s, v_i \rangle : s \in S\}$. Show that $S = P + Q \Leftrightarrow P, Q$ are defined by the same outer normals, $h(S, v_i) = h(P, v_i) + h(Q, v_i)$, $i = 1, \dots, n$, and neither of P, Q is a point.*

3 Polynomials

There is a theory of polynomials which aspires to exploit their sparseness, in other words understand their behavior when we take into account only their nonzero terms, i.e., terms for which the coefficient is known to be nonzero. Take a bivariate polynomial $f(x_1, x_2)$. Its *support* $A = \text{supp}(f) = \{a_1, \dots, a_\mu\} \subset \mathbb{N}^2$ is the set, with cardinality μ, of exponent vectors corresponding to terms with nonzero coefficients:

$$f(x_1, x_2) = c_1 x_1^{a_{11}} x_2^{a_{12}} + \cdots + c_\mu x_1^{a_{\mu 1}} x_2^{a_{\mu 2}}, \qquad c_j \neq 0,\ j = 1, \dots, \mu,$$

where $a_j = (a_{j1}, a_{j2}) \in \mathbb{N}^2$ is an exponent vector.

Definition 3.1 *The* Newton polygon *of $f(x_1, x_2)$ is the convex hull of the points in A.*

Figure 3 depicts the Newton polygon of a bivariate polynomial and compares it with the Newton polygon of a polynomial with the same total degree, where every coefficient is nonzero. Clearly, Newton polygons provide a more precise description than total degree does. The Newton polygon is named after Newton, who introduced the Newton diagram in studying bivariate polynomials[2]. The Newton diagram of points (j, g_j) was used for finding roots of real polynomials of degree > 4 in the lecture of October 7, 2014.

Problem 3.2 *Let f, g be bivariate polynomials with Newton polygons P, Q, respectively. Prove that the Newton polygon of $f \cdot g$ is the Minkowski sum $P + Q$.*

It follows that, if we can factorize a given bivariate polynomial into $f = g_1 \cdot g_2$, then it holds that $S = P_1 + P_2$, where S, P_1, P_2 are the Newton polygons of f, g_1, g_2, respectively. Hence, Minkowski decomposition provides a necessary condition for the factorization of a polynomial.

[2] I. Newton. *The Correspondence of I. Newton*, 1676-1687. Cambridge Univ. Press, UK, 1960.

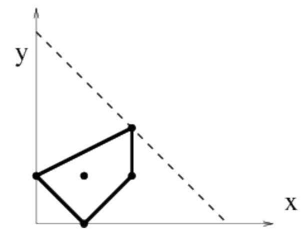

Figure 3: The Newton polygon of polynomial $c_1y + c_2x^2y^2 + c_3x^2y + c_4x + c_5xy$. The dotted triangle is the Newton polygon of the dense polynomial of the same total degree.

4 Mixed volume

This section considers volume (or area) of polygons and introduces the concept of mixed volume (or mixed area) defined for two polygons. We denote by $\text{Vol}(P)$ the Euclidean volume (area) of polygon P, where unit volume is assigned to the unit square; clearly, $\text{Vol}(P) = 0$ if P is a line segment or point.

We have defined addition between polygons. Let us now define scalar multiplication: The *scalar multiple* of any set $P \subset \mathbb{R}^2$ by a positive real number $\lambda \in \mathbb{R}_{>0}$ is $\lambda P = \{\lambda p \,|\, p \in P\} \subset \mathbb{R}^n$. In particular, if P is a polygon, so is λP. Moreover, we know the latter's volume: $\text{Vol}(\lambda P) = \lambda^2 \text{Vol}(P)$.

Equipped with these two operations, we can consider the following expression:

Proposition 4.1 *For given polygons P_1, P_2 and positive real parameters λ_1, λ_2, the expression* $\text{Vol}(\lambda_1 P_1 + \lambda_2 P_2)$ *is a homogeneous polynomial in λ_1, λ_2, of degree 2.*

Some of claims of this proposition are straightforward: Setting $\lambda_1 = 0$, the expression becomes $\text{Vol}(\lambda_2 P_2) = \lambda_2^2 \text{Vol}(P_2)$. By looking at examples, it is clear that there is also a multilinear term in the development of $\text{Vol}(\lambda_1 P_1 + \lambda_2 P_2)$, i.e., a term corresponding to product $\lambda_1 \lambda_2$. Hence,

$$\text{Vol}(\lambda_1 P_1 + \lambda_2 P_2) = \lambda_1^2 \text{Vol}(P_1) + \lambda_1 \lambda_2 M + \lambda_2^2 \text{Vol}(P_2),$$

for some $M \neq 0$. In fact $M \geq 0$.

Definition 4.2 *The coefficient of $\lambda_1 \lambda_2$ in* $\text{Vol}(\lambda_1 P_1 + \lambda_2 P_2)$ *is the* mixed volume $MV(P_1, P_2)$ *of P_1, P_2.*

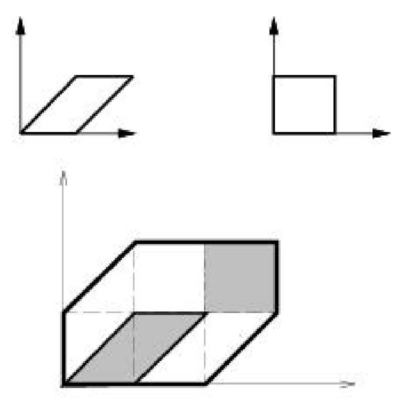

Figure 4: Two polygons, their Minkowski sum, and their mixed volume (white area in the Minkowski sum).

When $P_1 = P_2$ we set $\lambda_1 = \lambda_2$ and calculate $MV(P_1, P_2) = 2\text{Vol}(P_1)$. Mixed volume scales linearly with each λ_i. These lead to an equivalent definition of mixed volume:

Definition 4.3 *For polygons P_1, P_2, there is a unique, up to multiplication by scalar, real-valued function $MV(P_1, P_2) \geq 0$, called* mixed volume*, which is multilinear with respect to Minkowski addition and scalar multiplication:*

$$MV(P_1, \lambda P_2 + \rho P_2') = \lambda MV(P_1, P_2) + \rho MV(P_1, P_2'),\ \lambda, \rho \in \mathbb{R}_{\geq 0},\ \textit{polygon } P_2'.$$

To fully define mixed volume we require that $MV(P_1, P_1) = 2\,\mathrm{Vol}(P_1)$.

Problem 4.4 *Compute the mixed volume when both polygons are unit triangles S with vertices on the coordinate axes and scaled by d_1, d_2.*

Obviously MV is invariant under permutation of the polygons. It is also invariant under translations, and under rotations that preserve volume. It is also monotone w.r.t. inclusion: $P_1' \subset P_1 \Rightarrow MV(P_1', P_2) \leq MV(P_1, P_2)$.

There is an explicit Inclusion-exclusion formula: $MV(P_1, P_2) = \mathrm{Vol}(P_1 + P_2) - \mathrm{Vol}(P_1) - \mathrm{Vol}(P_2)$.

Besides mixed volume, another famous multilinear function is the determinant (and also the permanent) of a matrix: When a matrix column is scaled by λ, the determinant is scaled by λ, and when it is the sum of two column vectors, the determinant becomes the sum of two matrix determinants, each matrix containing one summand column.

Problem 4.5 *For segments with $\dim P_1 = \dim P_2 = 1$, their $MV(P_1, P_n) = \mathrm{Vol}(P_1 + P_2) = |\det P|$, where $P = [P_1\ P_2]$ is a 2×2 matrix whose columns correspond to P_1, P_2.*

Note $P_1 + P_2$ is a parallelogram, which degenerates to a segment iff P_1, P_2 are parallel. In other words, vectors $P_1, P_2 \in \mathbb{R}^2$ are linearly independent iff $MV(P_1, P_2) > 0$.

Problem 4.6 $MV(Q_1, Q_2) > 0 \Leftrightarrow \exists$ *segments* $E_i : \dim E_i = 1, E_i \subset Q_i,\ i = 1, 2$ *s.t.* $MV(E_1, E_2) > 0$.

4.1 Counting roots

We already saw how Newton polygons provide a bridge from the algebraic to the geometric setting, since Minkowski addition and decomposition are related to polynomial multiplication and factorization. There is an algebraic theory that considers the Newton polygons of bivariate (and multivariate) polynomials so as to obtain "algebraic" information on them such as the number of common roots for two bivariate polynomials.

The cornerstone of this theory is Bernstein's upper bound on the number of common roots of a square polynomial system[3]. This bound is also called the BKK after Kushnirenko and Khovanskii. We state it for two bivariate polynomials.

Theorem 4.7 *Consider bivariate polynomials f_1, f_2 with real coefficients. If their coefficients are generic (e.g., random), the number of their complex common roots is $MV(Q_1, Q_2)$, where Q_i is the Newton polygon of f_i.*

Problem 4.8 $c_0 + c_1x + c_2x^2y + c_3xy, b_0 + b_1x + b_2y + b_3xy$*. The Newton polygons and their Minkowski sum is shown in Figure 4. Compute the mixed volume: is it optimal? Also, compare it to the classic Bézout bound.*

Problem 4.9 *Show that BKK generalizes the fundamental theorem of algebra for a univariate polynomial. Show that it is at most as high as Bézout's bound, which equals the product of the polynomials' total degrees: for which Newton polygons the two bounds coincide?*

[3] D.N. Bernstein. The number of roots of a system of equations, *Function. Anal. & Applic.*, 9:183–185, 1975. Translated from the Russian.