Polynomial decompositions and applications

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Tensors

Tensors (also referred to as *multi-way arrays*) are present both in pure mathematics and in more applied sciences: computer science, engineering, physics, chemistry... The central problem I will consider here is about *tensor decomposition*. We have a set of tensors that are called *simple*. Given any tensor we would like to express it as a sum of simple tensors.

As an example, imagine a single antenna that receives a signal from many mobile phones at the same time. The receiver must decompose this superposed electromagnetic wave into original *simple* signals, each one encoding a single conversation.

Figure 1: More and more people use mobile phones. A lot of advanced mathematics is involved in wireless communication. A part of it is the theory of tensor decomposition.

As another example, fluorescence spectroscopy is a method to analyse samples of solutions and determine concentration of chemicals. Each sample is excited by light at various wavelengths and the light emitted is measured. The data is collected as a tensor, and we need to determine the decomposition of this tensor in order to extract information about the chemicals and their concentrations.

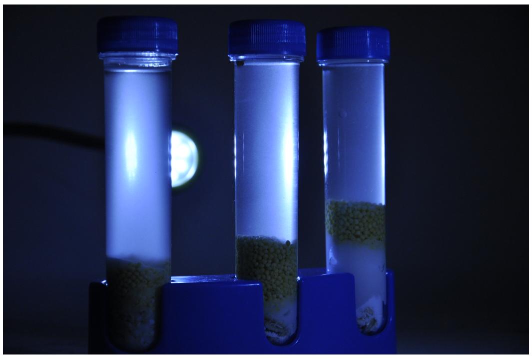

Figure 2: Mathematicians rather do not use spectroscopy. However we work on the theory of tensor decomposition involved. Here are some toys of my daughter. *Oats*, *potato starch* and *millet* represent chemicals with various fluorescent properties, and they are disolved in water in different concentrations.

In other words, for many sciences it is critical to extract simple and meaningful ingredients from some complicated data. In mathematical terms, this corresponds to the problem of decomposing tensors. The key notion here is *rank of tensor*.

The *rank of a tensor* is the minimal number of simple tensors that are needed in the decomposition. Thus in the examples above, it *should* correspond to the number of mobile phones talking, or the number of different chemicals in the sollutions. "Should correspond", not "corresponds", because there are situations when the decompositions are not unique. For example, if there are too many mobile phones for the capabilities of antenae, or too few samples with solutions containing too many different chemicals.

Polynomial decompositions

In this class we will study the decompositions of polynomials into sums of powers. Let $P(x,y)$ be a homogeneous polynomial in two variables:

$$P(x) = a_d x^d + a_{d-1} x^{d-1} y + \cdots + a_1 x y^{d-1} + a_0 y^d$$

where $a_0, a_1, \ldots, a_d$ are real (or complex) numbers.

Problem. *What is the minimal number r of linear polynomials $\ell_i(x) = b_i x + c_i y$, such that*

$$P(x,y) = \pm \ell_1^d \pm \ell_2^d \pm \cdots \pm \ell_r^d ?$$

Such number r is called the rank *of $P(x,y)$ (or Waring rank, or symmetric rank).*

Exercise 1. *Find the rank of* $P(x,y) = x^2 + xy + \frac{1}{2}y^2$.

The *derivatives* of $P(x,y)$ are the polynomials:

$$\alpha \lrcorner P := \frac{\partial P(x,y)}{\partial x} = d a_d x^{d-1} + (d-1)a_{d-1}x^{d-2}y + \cdots + 2a_2 x y^{d-2} + a_1 y^{d-1}$$
$$\beta \lrcorner P := \frac{\partial P(x,y)}{\partial y} = a_{d-1}x^{d-1} + 2a_{d-2}x^{d-2}y + \cdots + (d-1)a_1 x y^{d-2} + d a_0 y^{d-1}$$

Exercise 2. *Calculate* $\alpha \lrcorner (x^2 + xy + \frac{1}{2}y^2)$, $\beta \lrcorner (x^2 + xy + \frac{1}{2}y^2)$.

We can iterate derivatives: for instance, a *double derivative* of $P(x,y)$ in x is the polynomial

$$\alpha^2 \lrcorner P := d(d-1)a_d x^{d-2} + (d-1)(d-2)a_{d-1}x^{d-3}y + \cdots + a_2 y^{d-2}$$

and so on. The "hook $\lrcorner$" notation is convenient to allow algebraic operations on derivatives: for example,

$$\begin{aligned} (\alpha\beta - \alpha^2) \lrcorner P &= (\alpha \lrcorner (\beta \lrcorner P)) - \alpha \lrcorner (\alpha \lrcorner P) \\ (\alpha^2 + \alpha\beta - 2\beta^2) \lrcorner P &= \alpha \lrcorner (\alpha \lrcorner P) + (\alpha \lrcorner (\beta \lrcorner P)) - 2(\beta \lrcorner (\beta \lrcorner P)) P(x) \\ (\alpha - \beta)(\alpha + 2\beta) \lrcorner P &= \alpha \lrcorner (\alpha \lrcorner P + 2\beta \lrcorner P) - \beta \lrcorner (\alpha \lrcorner P + 2\beta \lrcorner P). \end{aligned}$$

Exercise 3. *Calculate*

$$(\alpha\beta - \alpha^2) \lrcorner (x^2 y - x y^2).$$

The following are the derivation rules:

- $\Theta \lrcorner (P + R) = \Theta \lrcorner P + \Theta \lrcorner R$;
- $(\Theta + \Phi) \lrcorner P = \Theta \lrcorner P + \Phi \lrcorner P$;
- $(f\Theta) \lrcorner P = \Theta \lrcorner (fP) = f(\Theta \lrcorner P)$;
- $(\Theta + \Phi) \lrcorner (P + R) = \Theta \lrcorner P + \Theta \lrcorner R + \Phi \lrcorner P + \Phi \lrcorner R$;
- $\Theta\Phi \lrcorner P = \Theta \lrcorner (\Phi \lrcorner P) = \Phi \lrcorner (\Theta \lrcorner P)$;
- $(f\alpha + g\beta) \lrcorner (PR) = ((f\alpha + g\beta) \lrcorner P)R + P((f\alpha + g\beta) \lrcorner R)$.

Here $P = P(x,y)$, and $R = R(x,y)$ are polynomials, while Θ, and Φ are algebraic expressions in α and β, for example $\Theta = \alpha^2 + \alpha\beta - 2\beta^2$, $\Phi = \alpha^3 - \beta^3$ and so on. f and g are (real or complex) numbers, and $(f\alpha + g\beta)$ is a linear derivative.

Exercise 4. *Show that* $(b\alpha - a\beta) \lrcorner \left[(ax + by)^d\right] = 0$ *for all numbers* a, b *and for any degree* $d > 0$.

Exercise 5. *Show that* $\beta(2\alpha-\beta)\lrcorner\left[(x+2y)^d+x^d\right]=0$ *for any degree* $d>0$.

Exercise 6. *Show that* $(\alpha+\beta)\beta(2\alpha-\beta)\lrcorner\left[(x+2y)^d+x^d+(x-y)^d\right]=0$ *for any degree* $d>0$.

Exercise 7. *Suppose that* $(b\alpha-a\beta)\lrcorner P=0$ *for some homogeneous polynomial* $P=P(x,y)$ *of degree* d. *Show that* $P=c(ax+by)^d$ *for some number* c.

Exercise 8. *Find a (non-zero) quadratic expression in* α *and* β *of the form*

$$\Theta=a\alpha^2+b\alpha\beta+c\beta^2,$$

such that $\Theta\lrcorner(x^3y+xy^3)=0$. *What is the rank of* $P(x,y)=x^3y+xy^3$*?*

General method of finding rank of $P(x,y)$: Find an algebraic expression Θ in α and β such that $\Theta\lrcorner P=0$, and

$$\Theta=(b_1\alpha-a_1\beta)(b_2\alpha-a_2\beta)\cdots(b_r\alpha-a_r\beta).$$

If the linear forms $(b_i\alpha-a_i\beta)$ are pairwise non-proportional, then we can write the following decomposition of P

$$P(x,y)=c_1(a_1x+b_1y)^d+c_2(a_2x+b_2y)^d+\cdots+c_r(a_rx+b_ry)^d$$

for some numbers c_r. The lowest possible degree r of such Θ is the rank of P, and in this case we say that the decomposition above is called a *minimal decomposition.*

Exercise 9. *Find the rank of* $P(x,y)=x^3y$. *Is the minimal decomposition of this polynomial unique (up to order and rescaling, see below)?*

Let me remark that we can always change the order in the decomposition, and we can also rescale the linear forms. For instance, let

$$P(x,y)=3x^6+6x^5y+75x^4y^2+140x^3y^3+255x^2y^4+186xy^5+65y^6.$$

Then

$$\begin{aligned}P(x,y)&=(x+2y)^6+x^6+(x-y)^6\\&=x^6+(x-y)^6+(x+2y)^6\\&=\tfrac{1}{729}(3x-3y)^6+64(\tfrac{1}{2}x+y)^6+x^6\end{aligned}$$

and so on. All these are considered the same decomposition up to order and rescaling. In fact, the polynomial P defined above has a unique decomposition (up to order and rescaling).

Exercise 10. *Prove that the rank of* $P(x,y) = x^4y + 8x^2y^3 + \frac{16}{5}y^5$ *is* 2. *Is there a unique minimal decomposition of this polynomial (up to order and rescaling)?*

Exercise 11. *Can you find a homogeneous polynomial* $P(x,y)$ *of rank* 2, *which has more than one decomposition as a sum of two powers of linear forms:*

$$P(x,y) = (a_1x + b_1y)^d + (a_2x + b_2y)^d?$$

How many such decompositions does your polynomial have (up to order and rescaling)?

More advanced material

Analogously, we can consider homogeneous polynomials in more variables and decompose them into sums of powers of linear forms.

Exercise 12. *Let* $P(x,y,z) = x^4 + y^4 + z^4 - 12x^2yz - 12xy^2z - 12xyz^2$. *Show* $(\alpha\beta - \alpha\gamma - \beta^2 + \gamma^2) \lrcorner P = 0$. *Here* γ *denotes the derivative with respect to* z.

The *algebraic expressions* in $\alpha, \beta, \gamma, \ldots$ really are just polynomials in these variables, although we have to remember their role as derivations of polynomials in $x, y, z, \ldots$ Denote by $\mathbb{C}[\alpha, \beta, \gamma, \ldots]$ the set of all polynomials in $\alpha, \beta, \gamma, \ldots$ with complex coefficients.

Let $\mathcal{A}nn(P) \subset \mathbb{C}[\alpha, \beta, \gamma, \ldots]$ be the *annihilator* of P:

$$\mathcal{A}nn(P) := \{\Theta \in \mathbb{C}[\alpha, \beta, \gamma, \ldots] \mid \Theta \lrcorner P = 0\}.$$

Exercise 13. *Show that if* $\Theta, \Phi \in \mathcal{A}nn(P)$, *and* $\Psi \in \mathbb{C}[\alpha, \beta, \gamma, \ldots]$, *then* $a\Theta + b\Phi \in \mathcal{A}nn(P)$, *and* $\Theta\Psi \in \mathcal{A}nn(P)$ *for any complex numbers* a, b.

This exercise shows $\mathcal{A}nn(P)$ is an *ideal* in the *polynomial ring*.

Exercise 14. *Let* $\Theta \in \mathbb{C}[\alpha, \beta, \gamma, \ldots]$ *be homogeneous of degree* d. *Let* $\ell = ax + by + cz + \cdots$ *be a linear polynomial in* x, y, z. *Prove that:*

$$\Theta(a, b, c, \ldots) = 0 \iff \Theta \lrcorner (\ell^d) = 0.$$

Here $\Theta(a, b, c, \ldots)$ *denotes the evaluation of* Θ *at* $\alpha = a$, $\beta = b$, $\gamma = c, \ldots$ *In the situation of equivalence, we say that* Θ *vanishes on* ℓ.

Exercise 15. *Suppose* $P = \ell_1^d + \ell_2^d + \cdots + \ell_r^d$, *and* $\Theta \in \mathbb{C}[\alpha, \beta, \gamma, \ldots]$ *is homogeneous. Show that if* Θ *vanishes on each* ℓ_i, *then* $\Theta \lrcorner P = 0$.

Exercise 16. *Let* $P(x,y,z) = x^4 + y^4 + z^4 - 12x^2yz - 12xy^2z - 12xyz^2$. *Find two non-proportional* $\Theta, \Phi \in \mathbb{C}[\alpha, \beta, \gamma]$ *of degree* 2 *such that*

$$\Theta \lrcorner P = \Phi \lrcorner P = 0.$$

What can you say about the rank of P*?*