SEQUENCES

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Warm up problems

(1) What is $\binom{11}{7}$?
(2) A polynomial $P(x)$ with integer coefficients satisfies $P(2) = 0$. Show that $P(2014)$ is even and that $P(5)$ is divisible by 3.
(3) A polynomial $P(x)$ with integer coefficients is such that there are four distinct numbers a, b, c, d with $P(a) = P(b) = P(c) = P(d) = 16$. Show that there is no n for which $P(n) = 5$.
(4) Let F_n be the Fibonacci sequence $1, 1, 2, 3, 5, \ldots$. Show that $|F_n^2 - F_{n-1}F_{n+1}| = 1$ for all positive integers n.
(5) Show that any positive integers can be written uniquely as a sum of non-consecutive Fibonacci numbers.
(6) n points are marked on a circle. The points are connected by lines lying inside the circle in such a way that no three lines intersect at a point in the interior of the circle. Into how many parts has the circle been divided? (solution at the end)
(7) In how many ways can you tile a $2 \times n$ rectangle by 2×1 dominoes?
(8) In how many ways can you tile a $2 \times n$ rectangle by 2×1 and 2×2 tiles?
(9) Choose a basic shape of varying size and a basic set of tiles, make up a similar problems as above, and solve it.
(10) Can you find a closed formula for the previous numbers?

Recurrence sequences

A recurrence sequence is a sequence $\mathbf{x} = (x_0, x_1, x_2, \ldots)$ defined by:

$$x_{n+k} = a_1 x_{n+k-1} + \ldots a_k x_n, \qquad x_0 = c_0, \ldots, x_{k-1} = c_k$$

where c_i and a_i are given.

Theorem. Any recurrence sequence x_n can be given in closed formula by

$$x_n = \sum_{i=0}^{m} A_i(n)\alpha_i^n$$

where the α_i are the solutions of $T^k - a_1 T^{k-1} - \ldots - a_{k-1}X - a_k = 0$ and the $A_i(n)$ are polynomials (computable from a_i, c_j).

Problem: Show that if x_n is a recurrence sequence then the power series

$$S(T) = \sum_{k=0}^{\infty} x_n T^n$$

is a rational function (ratio $P(T)/Q(T)$ where P and Q are polynomials in T).

Hint: show that $F(T)S(T)$ is a polynomial if $F(T) = 1 - a_1 T - \ldots - a_{k-1}T^{k-1} - a_k T^k$.

Zeroes of recurrence sequences

Find the zeroes of the following sequences: the Fibonacci sequence; the sequence $x_0 = 0, x_1 = 1, x_{n+2} = -x_n$; ...

Theorem (Skolem-Mahler-Lech). Let x_n be a recurrence sequence of integers. The set of n for which x_n is zero is the union of a finite set and finitely many arithmetic progressions; in these arithmetic progressions, the common difference depends only on the roots α_i.

Proof of the theorem, in the case where the roots α_i are integers. In this case $\alpha_i^{p-1} \equiv 1$ modulo p for any p which does not divide α_i. Choose one such p. Then we can consider different subsequences $y_m^{(i)} = x_{i+m(p-1)}$ according to the residue classes of the index n modulo $p-1$. If x_n is zero infinitely often then so is at least one of the $y_m^{(i)}$.

Now, dropping i, we can write

$$y_m = \sum_{j=1}^{r} B_j(m)(1+p\beta_j)^m$$

where B_j is a polynomial and $\beta_j = (\alpha_j^{p-1} - 1)/p$. Now we can write

$$B(m)(1+p\beta)^m = \sum_{k\geq 0} p^k P_k(m) = P(m)$$

where the P_k are polynomials in m, using the binomial theorem. So $P(m)$ does not have a degree (which should be a number independent of the variable m if it existed!) but modulo any p^k it is just a polynomial. If m_0 is a solution we can write $P(m) = (m - m_0)Q(m)$ where $Q(m) = \sum_{k\geq 0} p^k Q_k(m)$ and Q_0 has smaller degree than P_0, or $Q_0 = 0$. If there are infinitely many zeros then we can repeat this enough times until we get to a point where the new Q has $Q_0 = 0$ i.e. Q is divisible by p. Repeat again: then find that P is divisible by... p^∞! So $P = 0$.

Problems: 0. Fill in the details of the previous sketch of proof. 1. Write down any sequence of integers in which every positive number occurs infinitely many times. 2. Show that you cannot do that with a recurrence sequence of integers.

Solution to (6): the formula is

$$1 + \binom{n}{2} + \binom{n}{4}.$$

It was found using a recursion (not of the *linear* type discussed above) suggested independently by two students. Problem: can you show it is the right formula directly without using a recursion?

If you have any comments on all this feel free to email me.