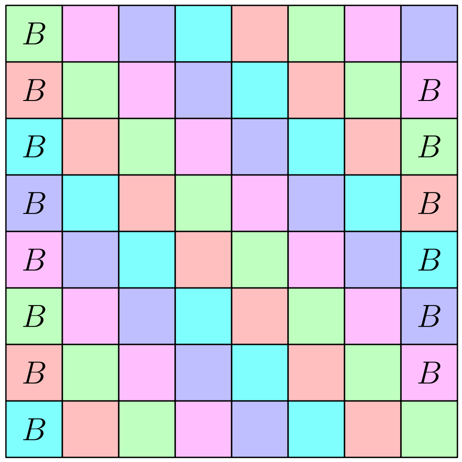
Berkeley Math Circle
Monthly Contest 7, Solutions

1. Consider an 8×8 chessboard, on which we place some *bishops* in the 64 squares. Two bishops are said to attack each other if they lie on a common diagonal.

   (a) Prove that we can place 14 bishops in such a way that no two attack each other.

   (b) Prove that we cannot do so with 15 bishops.

   Solution. For the first part, here is one maximal arrangement, where the location of the bishops are indicated by the letter B.



   To see that there cannot be 15 bishops, observe that we have highlighted 15 right-down diagonals in the square above. Each diagonal can accommodate at most one bishop. Furthermore, the lower-left corner and the upper-right corner constitute diagonals of size 1 which cannot be both occupied. This gives the bound of 14 bishops.

2. As usual, let $n!$ denote the product of the integers from 1 to n inclusive. Determine the largest integer m such that $m!$ divides $100! + 99! + 98!$.

   Solution. The answer is $m = 98$. Set

$$N = 98! + 99! + 100! = 98! \left(1 + 99 + 99 \cdot 100\right).$$

   Hence N is divisible by $98!$. But

$$\frac{N}{98!} = 1 + 99 \cdot 101$$

   is not divisible by 99. Hence N is not divisible by $99!$.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(x + y) = f(x - y) + 4xy$$

   for all real numbers x and y.

   Solution. The answer is the functions $f(x) = x^2 + c$, where c is a constant. It is easy to check that all such functions work, since

$$(x + y)^2 + c = (x - y)^2 + c + 4xy$$

   is trivially true.

   Now, we prove these are the only functions. Put $x = y = \frac{1}{2}a$ to obtain

$$f(a) = f(0) + a^2$$

   for all real numbers a, which implies the conclusion.

4. (a) Let $WXYZ$ be a rectangle, and locate points A, B, C, D on the sides WX, XY, YZ, ZW such that

$$\frac{WA}{AX} = \frac{XB}{BY} = \frac{YC}{CZ} = \frac{ZD}{DW} = 1 + \sqrt{2}.$$

Show that $ABCD$ is a parallelogram with $AB/BC = AC/BD$.

(b) Show that, conversely, any parallelogram such that the ratio of two adjacent sides equals the ratio of the diagonals can be obtained from a rectangle in this way.

Solution. (a) By construction, A and C are reflections about the center O of $WXYZ$, as are B and D. Hence $ABCD$ is a parallelogram. Let E be the foot of the perpendicular from A to YZ. Note that

$$\frac{AE}{BX} = \frac{XY}{BX} = \frac{2+\sqrt{2}}{1+\sqrt{2}} = \sqrt{2}$$

and

$$\frac{CE}{AX} = \frac{ZY - 2ZC}{ZC} = \sqrt{2}.$$

Hence triangles ACE and BAX are similar with ratio $\sqrt{2} : 1$, and thus $AC = \sqrt{2} \cdot AB$. Similarly $BD = \sqrt{2} \cdot BC$, and thus $AC/BD = AB/BC$.

(b) Conversely, let $ABCD$ be a parallelogram with center O, and suppose that $AB/BC = AC/BD$. The triangles ABC and AOB share the angle CAB and the side ratio $AB/BC = AO/OB$. Unfortunately, this is an SSA condition, but we can still conclude that the triangles are similar as follows. Draw $\triangle ABC'$ similar to $\triangle AOB$ with C' on ray AC; then C and C' are equidistant from B, and if they do not coincide, then we get $\angle BCC' = \angle BC'C$ and hence

$$180 = \angle BCA + \angle BC'A = \angle BCA + \angle OBA < \angle BCA + \angle CBA < 180,$$

a contradiction. So $\triangle ABC \sim \triangle AOB$. Since the area of the former triangle is twice the area of the latter, the ratio of similarity is $\sqrt{2} : 1$.

Next, we reconstruct the rectangle $WXYZ$ by intersecting the exterior angle bisectors ℓ_1, ℓ_2, ℓ_3, ℓ_4 of $\angle BAC$, $\angle CBD$, $\angle DCA$, and $\angle ADB$. For instance, the calculation

$$\begin{aligned}
\alpha + \beta &= 90 - \frac{\angle CAB}{2} + 90 - \frac{CBD}{2} - \angle DBA \\
&= 180 - \frac{(\angle CAB + \angle DBA) + (\angle CBD + \angle DBA)}{2} \\
&= 180 - \frac{180 - \angle AOB + \angle ABC}{2} = 90
\end{aligned}$$

shows that ℓ_1 and ℓ_2 meet perpendicularly at a point X. Since by symmetry $\ell_1 \parallel \ell_3$, $\ell_2 \parallel \ell_4$, we get that $WXYZ$ is a rectangle.

Finally, to compute the ratio WA/AX, drop the perpendicular E from A to YZ. We have $\triangle ACE \sim \triangle DCZ$ since ZY externally bisects $\angle DCA$, and the ratio of similarity is $AC/CD = \sqrt{2}$. Hence

$$\frac{WA}{AX} = \frac{CE + CZ}{CZ} = 1 + \sqrt{2},$$

and $XB/BY = YC/CZ = ZD/DW = 1 + \sqrt{2}$ symmetrically.

5. Prove that there exists an infinite set S of positive integers with the property that if we take any finite subset T of S, the sum of the elements of T is not a perfect kth power for any $k \geq 2$.

Solution. Consider the set

$$S = \{2,\ 2^2 \cdot 3,\ 2^2 \cdot 3^2 \cdot 5,\ 2^2 \cdot 3^2 \cdot 5^2 \cdot 7,\ 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11, \dots\}.$$

Let $T \subseteq S$ be a finite nonempty set, and m its smallest element. Let p be the prime factor which appears exactly once in m. Then p divides every element of T, and p^2 divides every element of T except m. From this it follows that the sum of the elements of T has exactly one factor of p; thus, it cannot be a kth power.

6. The numbers 1, 2, …, 2014 are arranged evenly around a circle in arbitrary order. We are permitted to swap two adjacent numbers, as long as they do not sum to 2015. Prove that it is impossible to perform finitely many swaps so that each number ends up diametrically opposite from its starting point.

Solution. Assume for contradiction it's impossible. Each time a number moves one slot counterclockwise, assign a score of $+1$; each time a number moves one slot clockwise, assign a score of -1. Thus every move produces a total score of $1 + (-1) = 0$.

Let $S(a)$ denote the total score accumulated by moves of a. Observe that if $a + b = 2014$, we see that in fact $S(a) = S(b)$. Thus

$$0 = \sum_{a=1}^{2014} S(a) = \sum_{a=1}^{1007} S(a) + S(2014 - a) = 2\sum_{a=1}^{1007} S(a).$$

Hence

$$0 = \sum_{a=1}^{1007} S(a).$$

On the other hand, since a is supposed to end up diametrically opposite its starting point, we are supposed to have $S(a) \equiv 1007 \pmod{2014}$ for every a. This gives

$$0 \equiv 1007 \cdot 1007 \equiv 1007 \pmod{2014}$$

which is impossible.

7. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions. Is it possible that $f(g(x))$ is strictly decreasing but $g(f(x))$ is strictly increasing?

Solution. First, divide the positive reals into alternating intervals, as follows: set

$$A = \bigcup_{k \in \mathbb{Z}} \left(2^{2k}, 2^{2k+1}\right]$$

and

$$B = \bigcup_{k \in \mathbb{Z}} \left(2^{2k+1}, 2^{2k+2}\right].$$

Thus A and B are disjoint sets, and moreover $A \cup B$ encompass all positive real numbers.

The point of this construction is so that for any $a \in A$ and $b \in B$, we have $2a \in B$ and $2b \in A$. This allows us to define the following tricky "alternating" function f: set

$$f(x) = \begin{cases} x & |x| \in A \\ -x & |x| \in B \\ 0 & x = 0. \end{cases}$$

This allows us to take $g(x) = 2f(x)$. Then $f(g(x)) = -2x$ is strictly decreasing, while $g(f(x)) = 2x$ is strictly increasing.