Berkeley Math Circle
Monthly Contest 7
Due April 7, 2015

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 7, Problem 3
Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Consider an 8×8 chessboard, on which we place some *bishops* in the 64 squares. Two bishops are said to attack each other if they lie on a common diagonal.
   (a) Prove that we can place 14 bishops in such a way that no two attack each other.
   (b) Prove that we cannot do so with 15 bishops.
2. As usual, let $n!$ denote the product of the integers from 1 to n inclusive. Determine the largest integer m such that $m!$ divides $100! + 99! + 98!$.
3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy
   $$f(x + y) = f(x - y) + 4xy$$
   for all real numbers x and y.
4. (a) Let $WXYZ$ be a rectangle, and locate points A, B, C, D on the sides WX, XY, YZ, ZW such that
   $$\frac{WA}{AX} = \frac{XB}{BY} = \frac{YC}{CZ} = \frac{ZD}{DW} = 1 + \sqrt{2}.$$
   Show that $ABCD$ is a parallelogram with $AB/BC = AC/BD$.
   (b) Show that, conversely, any parallelogram such that the ratio of two adjacent sides equals the ratio of the diagonals can be obtained from a rectangle in this way.
5. Prove that there exists an infinite set S of positive integers with the property that if we take any finite subset T of S, the sum of the elements of T is not a perfect kth power for any $k \geq 2$.
6. The numbers 1, 2, ..., 2014 are arranged evenly around a circle in arbitrary order. We are permitted to swap two adjacent numbers, as long as they do not sum to 2015. Prove that it is impossible to perform finitely many swaps so that each number ends up diametrically opposite from its starting point.
7. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions. Is it possible that $f(g(x))$ is strictly decreasing but $g(f(x))$ is strictly increasing?