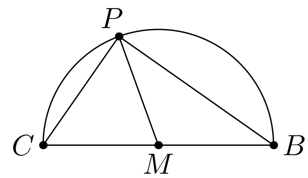
Berkeley Math Circle
Monthly Contest 6, Solutions

1. Let $ABCD$ be a square. We randomly select a point P inside the square, uniformly and at random. What is the probability that $\angle BPC > 90°$?

   Solution. Let ω be the semicircle whose diameter is $\overline{BC}$ and whose interior lies inside $ABCD$. Let M be denote the center of ω. We use the result that

   - $\angle BPC < 90°$ if P is inside ω,
   - $\angle BPC = 90°$ if P is on ω, and
   - $\angle BPC > 90°$ if P is outside ω.

   In order to prove this, we first show the result when P is on ω (this result is sometimes called *Thales' Theorem.*)

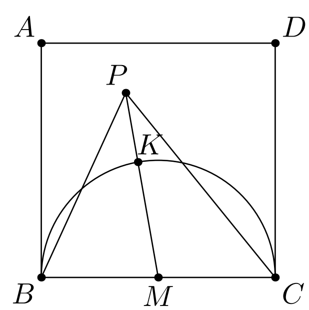


   Compute

$$\begin{aligned} 180° &= 360° - (\angle PMB + \angle PMC) && \text{as } \angle PMB + \angle PMC = 180° \\ &= (180° - \angle PMB) + (180° - \angle PMC) && \\ &= 2\angle BPM + (180° - \angle PMC) && MB = MP \text{ in } \triangle BMP \\ &= 2\angle BPM + 2\angle CPM && MC = MP \text{ in } \triangle CMP \\ &= 2\angle BPC \end{aligned}$$

   as claimed.

   In the case that P is outside ω, we may intersect segment MP with ω to obtain a point K. Then $90° = \angle BKC > \angle BPC$ (since, say $\angle BKM > \angle BPM$ and $\angle CKM > \angle CPM$).



   The case where P is inside ω is dealt in the same way.

   Thus, we see that we seek the probability that P lies inside ω. The area of ω inside $ABCD$ is simply $\frac{1}{2}\pi(\frac{1}{2})^2 = \frac{1}{8}\pi$ (choosing our units so that $AB = 1$), and the area of $ABCD$ is 1; hence the answer is $\frac{1}{8}\pi$.

2. Alice picks four numbers from the set $\{1, 2, 3, 4, 5, 6\}$, tells Bob their product and asks him to guess their sum. Bob realizes he cannot even determine for sure whether the sum is odd or even. What is the product of the numbers Alice chose?

Solution. Let P be said product. Evidently there are two distinct sets of numbers $\{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3, y_4\}$ such that $x_1x_2x_3x_4 = y_1y_2y_3y_4 = P$, but $x_1 + x_2 + x_3 + x_4$ and $y_1 + y_2 + y_3 + y_4$ have different parity.

Instead of considering the four numbers Alice picks, we consider instead the two numbers Alice does *not* pick; $\{x_5, x_6\}$ and $\{y_5, y_6\}$. They have the same properties we described above, since

$$x_5x_6 = \frac{6!}{x_1x_2x_3x_4} = \frac{720}{P} = \frac{6!}{y_1y_2y_3y_4} = y_5y_6$$

and analogously the sums also have different parities.

Thus we are looking for pairs of distinct numbers in $\{1, 2, 3, 4, 5, 6\}$ which have the same product but different sums. We can record the entire multiplication table, as below.

$\times$	1	2	3	4	5	6
1		2	3	4	5	6
2			6	8	10	12
3				12	15	18
4					20	24
5						30
6						

The numbers with duplicate entries are

$$\begin{aligned} 6 &= 6 \cdot 1 = 2 \cdot 3 \\ 12 &= 2 \cdot 6 = 3 \cdot 4 \end{aligned}$$

Of these, 12 has the desired property but 6 does not. Hence, Alice chose one of the quadruples $\{1, 3, 4, 5\}$ or $\{1, 2, 5, 6\}$; the product is then $P = 1 \cdot 3 \cdot 4 \cdot 5 = 1 \cdot 2 \cdot 5 \cdot 6 = 60$.

3. (a) Prove that for all real numbers x and y,

$$x^2 - 2y^2 = -[(x + 2y)^2 - 2(x + y)^2].$$

(b) How many positive integer solutions does the equation $x^2 - 2y^2 = 1$ have?

(c) How many positive integer solutions does the equation $x^2 - 2y^2 = 5$ have?

Remark. When we ask "how many," we ask for an answer (either a nonnegative integer, or that there are infinitely many) with proof.

Solution. (a) This is simply an algebra calculation:

$$\begin{aligned} -[(x + 2y)^2 - 2(x + y)^2] &= -[x^2 + 4xy + 4y^2 - 2(x^2 + 2xy + y^2)] \\ &= -[x^2 + 4xy + 4y^2 - 2x^2 - 4xy - 2y^2] \\ &= -[-x^2 + 2y^2] \\ &= x^2 - 2y^2. \end{aligned}$$

(b) There are infinitely many. Begin at the solution $x_0 = 3$, $y_0 = 2$ and apply the transformation

$$x_{n+1} = x_n + 2y_n, \quad y_{n+1} = x_n + y_n.$$

By part (a), the value of $x_n^2 - 2y_n^2$ is alternately 1 and -1 (to be precise, $(-1)^n$). Also, all the x_n are positive so $y_0 < y_1 < y_2 < \cdots$ and we never repeat a pair (x, y). Thus we get infinitely many solutions to $x^2 - 2y^2 = 1$. (With some more work one can prove that the (x_{2n}, y_{2n}) are *all* positive integer solutions.)

(c) There are no solutions. Look at the remainders of the terms mod 5. The term x^2 must be 0, 1, or 4; the term $2y^2$ must be 0, 2, or 3. The only way these can equal each other is if $5|x$ and $5|y$. But then $25|x^2 - 2y^2 = 5$, a contradiction.

4. The Moria Indestructible Phone Co. has hired you to test the hardiness of their newest smartphone model, the Mithril II. Your assignment is to determine the lowest floor of the Burj Khalifa tower (the world's tallest building, with 163 floors) from which the phone must be dropped to break it. You can ride the elevator to any floor, drop the phone to the ground, and then test whether it is intact. You may assume that if the phone breaks at a given floor, it consistently breaks at that floor and all higher floors. But the company has given you only two Mithril II's to test, and once one of them breaks, it remains broken.

   What is the minimum number of drops needed to determine the minimum floor of breaking, or else to conclude that the phone will withstand dropping from any of the floors?

   Solution. Here is a strategy requiring at most 18 drops. Drop the first phone from the 18th floor. If it breaks, drop the second phone from floors $1, 2, \ldots, 17$ in that order to determine the minimum breaking floor. Otherwise, drop the first phone from the $18 + 17 = 35$th floor. If it breaks, use the 16 remaining drops to test the second phone on floors 19 through 34. Otherwise, drop the first phone from the $18 + 17 + 16$th floor, and so on. If the first phone makes it to the $18 + 17 + \cdots + 5 = 161$st floor, then there are 4 drops left, more than enough to test it on the last two floors.

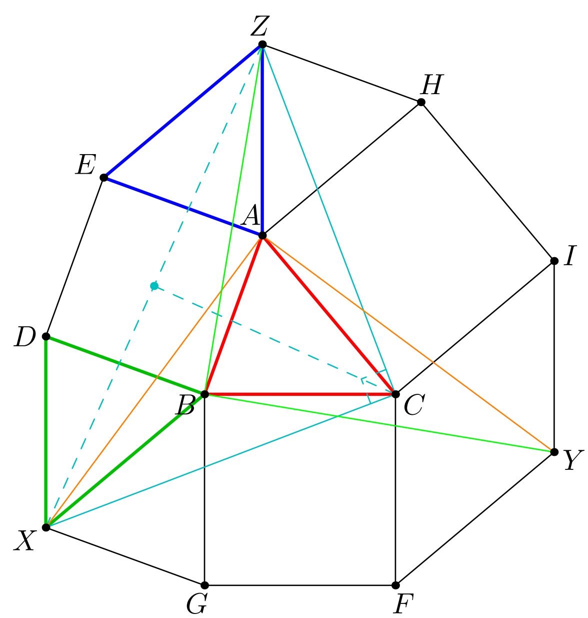
   Suppose that there existed a strategy requiring at most 17 drops. There are 164 possible "strengths" of the Mithril II (the breaking floor could be $1, 2, \ldots, 163$ or undefined). When the strategy is applied to one of these strengths, the result is a sequence of intact (I) or broken (B) outcomes containing at most two B's and having length at most 17. In fact, we can arrange for the sequence to have length exactly 17 by declaring that, if at a given point the strategy stops and declares the phone strength, the outcomes of any unused drops are arbitrarily designated I. Then each phone strength has a "signature" of 17 letters, each B or I, with at most two B's. But the number of possible signatures is only

$$\binom{17}{0} + \binom{17}{1} + \binom{17}{2} = 154.$$

   This shows that there are some two phone strengths that have the same signature and therefore cannot be distinguished by the claimed strategy.

5. Squares $ABDE$, $BCFG$ and $CAHI$ are drawn exterior to a triangle ABC. Parallelograms $DBGX$, $FCIY$ and $HAEZ$ are completed. Prove that $\angle AYB + \angle BZC + CXA = 90°$.

   Solution. Let ρ be the $90°$ rotation about the center of square $ABDE$, counterclockwise (orienting $\triangle ABC$ to have its vertices in counterclockwise order). Note that segments CA and ZE are congruent and perpendicular (thanks to square $CAHI$ and parallelogram $HAEZ$), so $\rho(C) = Z$. Likewise, segments BC and DX are congruent and perpendicular, implying $\rho(X) = C$. Now $\rho(XC) = CZ$ which implies $\angle ZCX = 90°$. Likewise $\angle XAY = \angle YBZ = 90°$. With three of the angles of the reentrant hexagon $XAYBZC$ known, the sum of the other three is readily computed:



$$\begin{aligned}
&\angle AYB + \angle BZC + \angle CXA \\
&= (180° - \angle BAY - YBA) + (180° - \angle CBZ - \angle ZCB) + (180° - \angle ACX - \angle XAC) \\
&= 540° - (\angle BAY + \angle XAC) - (\angle CBZ + \angle YBA) - (\angle ACX + \angle ZCB) \\
&= 540° - (\angle XAY + \angle BAC) - (\angle YBZ + \angle CBA) - (\angle ZCX + \angle ACB) \\
&= 540° - 3 \cdot 90° - (\angle BAC + \angle CBA + \angle ACB) \\
&= 540° - 270° - 180° = 90°.
\end{aligned}$$

6. Let a, b, c be positive real numbers. Show that

$$\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}+9\sqrt[3]{abc}\le 4(a+b+c).$$

Solution. By the AM-GM inequality, we know that

$$(abc)^2 \le \frac{a^6+b^6+c^6}{3}.$$

Letting $x=\sqrt{\frac{a^6+b^6+c^6}{3}}\ge abc$ so that $3x^2=a^6+b^6+c^6$, we see that it suffices to show

$$x \le \frac{3x^2+15}{12}-\frac{3}{3x^2+3}=\frac{x^2+5}{4}-\frac{1}{x^2+1}.$$

Now we can compute

$$\begin{aligned}\frac{x^2+5}{4}-\frac{1}{x^2+1}-x &= \frac{(x^2+5)(x^2+1)-4}{4(x^2+1)}-x\\ &= \frac{(x^2+5)(x^2+1)-4-4x(x^2+1)}{4(x^2+1)}\\ &= \frac{x^4-4x^3+6x^2-4x+1}{4(x^2+1)}\\ &= \frac{x^4-4x^3+6x^2-4x+1}{4(x^2+1)}\\ &= \frac{(x-1)^4}{4(x^2+1)}\\ &\ge 0\end{aligned}$$

and we're done.

7. Decide whether there exist positive integers a, b, c such that $3(ab+bc+ca)$ divides $a^2+b^2+c^2$.

Solution. The answer is *no*: such integers do not exist. In what follows, $\nu_p(n)$ will denote the exponent of p in the prime factorization of n.

Assume without loss of generality that a, b, c do not have some common divisor, and $a^2+b^2+c^2=3k(ab+bc+ca)$. Write

$$(3k+2)(a^2+b^2+c^2)=3k\cdot(a+b+c)^2.$$

Since $3k+2\equiv 2 \pmod 3$, there is a prime $p\equiv 2 \pmod 3$ with $\nu_p(3k+2)$ odd (in particular, $p \mid 3k+2$).

We first show that $p\neq 2$. Let us assume on the contrary that $\nu_2(3k+2)$ is odd (in particular, k is even). Remark that since a,b,c are not all even, $\nu_2(a^2+b^2+c^2)\le 1$. Furthermore,

$$\nu_2(a+b+c)=0 \iff \nu_2(a^2+b^2+c^2)=0.$$

Now we consider two cases.

- Assume $\nu_2(k)\ge 2$. Then $\nu_2(3k+2)=1$, $\nu_2(a^2+b^2+c^2)\le 1$. Therefore

  $$\nu_2(k)+2\nu_2(a+b+c)\ge 2\ge \nu_2(3k+2)+\nu_2(a^2+b^2+c^2)$$

  but equality cannot occur since the relations

  $$\nu_2(a+b+c)=0 \quad\text{and}\quad \nu_2(a^2+b^2+c^2)=1$$

  cannot hold simultaneously.
- Assume $\nu_2(k)=1$. Then $\nu_2(3k+2)>1$ and is odd. Now

  $$\nu_2(3k+2)+\nu_2(a^2+b^2+c^2)=1+2\nu_2(a+b+c).$$

  Thus $\nu_2(a^2+b^2+c^2)$ must be even, so it is zero; consequently $\nu_2(a+b+c)=0$ as well and we obtain $1<\upsilon_2(3k+2)=1$.

Now for the interesting part. Remark $p \mid a+b+c$ and $p \mid a^2+b^2+c^2$. Without loss of generality $b \not\equiv 0 \pmod{p}$, so that

$$a^2 + b^2 + (a+b)^2 \equiv 0 \pmod{p} \implies a^2 + ab + b^2 \equiv 0 \pmod{p}.$$

Then if $x = ab^{-1}$, we get that $x^2 + x + 1 \equiv 0 \pmod{p}$. But the left-hand side is the third cyclotomic polynomial, so either $p = 3$ or $3 \mid p - 1$, but neither is the case.

Therefore, such a triple (a, b, c) does not exist.