Berkeley Math Circle
Monthly Contest 6
Due March 3, 2015

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 6, Problem 3
Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Let $ABCD$ be a square. We randomly select a point P inside the square, uniformly and at random. What is the probability that $\angle BPC > 90°$?

2. Alice picks four numbers from the set $\{1, 2, 3, 4, 5, 6\}$, tells Bob their product and asks him to guess their sum. Bob realizes he cannot even determine for sure whether the sum is odd or even. What is the product of the numbers Alice chose?

3. (a) Prove that for all real numbers x and y,

$$x^2 - 2y^2 = -[(x + 2y)^2 - 2(x + y)^2].$$

   (b) How many positive integer solutions does the equation $x^2 - 2y^2 = 1$ have?

   (c) How many positive integer solutions does the equation $x^2 - 2y^2 = 5$ have?

   Remark. When we ask "how many," we ask for an answer (either a nonnegative integer, or that there are infinitely many) with proof.

4. The Moria Indestructible Phone Co. has hired you to test the hardiness of their newest smartphone model, the Mithril II. Your assignment is to determine the lowest floor of the Burj Khalifa tower (the world's tallest building, with 163 floors) from which the phone must be dropped to break it. You can ride the elevator to any floor, drop the phone to the ground, and then test whether it is intact. You may assume that if the phone breaks at a given floor, it consistently breaks at that floor and all higher floors. But the company has given you only two Mithril II's to test, and once one of them breaks, it remains broken.

   What is the minimum number of drops needed to determine the minimum floor of breaking, or else to conclude that the phone will withstand dropping from any of the floors?

5. Squares $ABDE$, $BCFG$ and $CAHI$ are drawn exterior to a triangle ABC. Parallelograms $DBGX$, $FCIY$ and $HAEZ$ are completed. Prove that $\angle AYB + \angle BZC + CXA = 90°$.

6. Let a, b, c be positive real numbers. Show that

$$\frac{a^6 + b^6 + c^6 + 15}{12} - \frac{3}{a^6 + b^6 + c^6 + 3} \geq abc.$$

7. Decide whether there exist positive integers a, b, c such that $3(ab + bc + ca)$ divides $a^2 + b^2 + c^2$.