Berkeley Math Circle
Monthly Contest 5, Solutions

1. Determine the number of ways to fill a 3×3 grid with 0's and 1's such that each row and column has an even sum.

   Solution. The answer is $2^4 = 16$. Denote the table as follows:

a_1	a_2	a_3
b_1	b_2	b_3
c_1	c_2	c_3

   Observe that the condition is equivalent to saying that every entry in the table has the same parity as the sum of the entries in its row (resp. column).

   We claim that upon filling a_1, a_2, b_1, b_2 there is exactly one way to fill the table. Indeed, we require that

   - a_3 has the same parity as $a_1 + a_2$,
   - b_3 has the same parity as $b_1 + b_2$,
   - c_1 has the same parity as $a_1 + b_2$,
   - c_2 has the same parity as $a_2 + b_2$.

   All that's left to do is check that there is exactly one way to fill in the entry c_3. We require c_3 to have the same parity as $a_3 + b_3$. But the parity of this is the same as that of $a_1 + a_2 + b_1 + b_2$. Similarly, $c_1 + c_2$ has the same parity as $a_1 + b_1 + a_2 + b_2$. Hence there is exactly one way to fill in c_3.

   Thus the claim is proved, and the number of ways to fill the entire table is merely the number of ways to fill the 2×2 upper-left square, which is $2^4 = 16$ as desired.

   Remark. This solution generalizes to establish that in general, for an $n \times n$ grid (instead of a 3×3 grid) the answer is $2^{(n-1)^2}$. By using the notation of "taking modulo 2" one can shorten the above solution significantly.

2. Prove that there exists a polynomial $f(x, y, z)$ with the following property: the numbers $|x|$, $|y|$, and $|z|$ are the sides of a triangle if and only if $f(x, y, z) > 0$.

   Solution. Answer:

$$f(x,y,z) = (x+y+z)(-x+y+z)(x-y+z)(x+y-z) \quad [= x^2y^2 + y^2z^2 + z^2x^2 - x^4 - y^4 - z^4].$$

   It is easily seen that the transformation $x \mapsto -x$, and symmetrically $y \mapsto -y$ and $z \mapsto -z$, do not change f, so it is enough to prove the following statement: *If x, y, and z are nonnegative reals, then x, y, and z are the sides of a triangle if and only if $f(x, y, z) > 0$.*

   Moreover, changing the order of x, y, and z does not change f, so we may assume that $x \le y \le z$. Now three of the factors of $f(x, y, z)$, namely $x + y + z$, $-x + y + z$, and $x - y + z$, are clearly nonnegative.

   If x, y, and z are the sides of a triangle, the familiar *triangle inequality* $x + y \ge z$ implies that the fourth factor $x + y - z$ is positive. Also, a side of a triangle cannot be zero, from which we get $x + y + z > 0$, $-x + y + z > 0$, $x - y + z > 0$, and hence $f(x, y, z) > 0$.

   Conversely, if $f(x, y, z) > 0$, then the four factors must be positive, so x, y, and z are positive and the triangle inequality $x + y > z$ holds. To construct the triangle, we may draw two circles of radii x and y whose centers Y, X are a distance z apart. Because each circle passes both inside and outside the other, the circles intersect at two points. Let Z be one. Then XYZ is the desired triangle.

3. A *triangular number* is one of the numbers $1, 3, 6, 10, 15, \ldots$ of the form $T_n = 1 + 2 + \cdots + n$ or, equivalently, $T_n = (n^2 + n)/2$.

   Find, with proof, all ways of writing 2015 as the difference of two triangular numbers.

   Solution. Since

$$T_m - T_n = \frac{m^2+m}{2} - \frac{n^2+n}{2} = \frac{m^2-n^2+m-n}{2} = \frac{(m+n+1)(m-n)}{2},$$

any expression of 2015 as the difference of two triangular numbers yields a factorization $2 \cdot 2015 = 4030 = (m+n+1)(m-n)$. Conversely, assigning values $a = m+n+1$, $b = m-n$ to the factors (necessarily both positive) yields a unique solution

$$m = \frac{a+b-1}{2} \qquad n = \frac{a-b-1}{2}$$

for m and n, which are positive integers if the following two conditions are satisfied:

- a and b have unlike parity (that is, one is even and one is odd). Since 4030 has exactly one prime factor of 2, this is no restriction.
- $a \geq b+2$. In particular, out of every pair of factors, we must put the larger factor for a.

The primes in the factorization $4030 = 2 \cdot 5 \cdot 13 \cdot 31$ may be apportioned to two factors in 16 ways, or 8 ways if the order of the factors is insignificant. So we get eight solutions:

a	b	Expression
$2 \cdot 5 \cdot 13 \cdot 31$	1	$T_{2015} - T_{2014}$
$5 \cdot 13 \cdot 31$	2	$T_{1008} - T_{1006}$
$2 \cdot 13 \cdot 31$	5	$T_{405} - T_{400}$
$13 \cdot 31$	$2 \cdot 5$	$T_{206} - T_{196}$
$2 \cdot 5 \cdot 31$	13	$T_{161} - T_{148}$
$5 \cdot 31$	$2 \cdot 13$	$T_{90} - T_{64}$
$2 \cdot 5 \cdot 13$	31	$T_{80} - T_{49}$
$5 \cdot 13$	$2 \cdot 31$	$T_{63} - T_1$

4. Let $ABCD$ be a quadrilateral whose diagonals are perpendicular and intersect at P. Let h_1, h_2, h_3, h_4 be the lengths of the altitudes from P to AB, BC, CD, DA. Show that

$$\frac{1}{h_1^2} + \frac{1}{h_3^2} = \frac{1}{h_2^2} + \frac{1}{h_4^2}. \tag{1}$$

Solution. The area of triangle ABP is equal to $h_1 \cdot AB/2$ and also to $AP \cdot BP/2$. Hence

$$\frac{1}{h_1^2} = \frac{AB^2}{AP^2 \cdot BP^2} = \frac{AP^2 + BP^2}{AP^2 \cdot BP^2} = \frac{1}{BP^2} + \frac{1}{AP^2}.$$

Applying the same transformation to all the terms of (1) yields

$$\left(\frac{1}{AP^2} + \frac{1}{BP^2}\right) + \left(\frac{1}{CP^2} + \frac{1}{DP^2}\right) = \left(\frac{1}{BP^2} + \frac{1}{CP^2}\right) + \left(\frac{1}{DP^2} + \frac{1}{AP^2}\right),$$

a triviality.

5. Weighted coins numbered $2, 3, 4, \ldots, 2015$ are tossed. The coin numbered i comes up heads with probability $1/(2i^2)$. What is the probability that an odd number of coins come up heads?

Solution. Let P_n be the probability that an odd number out of the coins whose numbers are at most n ($1 \leq n \leq 2015$ come up heads. For $2 \leq n \leq 2015$, there are two ways for this to happen: coin n is tails and an odd number of the preceding coins are heads, or coin n is heads and an even number of the preceding coins are heads. So we have a recursion

$$\begin{aligned} P_n &= \left(1 - \frac{1}{2n^2}\right) P_{n-1} + \frac{1}{2n^2}(1 - P_{n-1}) \\ &= \frac{2n^2 P_{n-1} - 2P_{n-1} + 1}{2n^2} \\ &= \frac{2(n^2-1)P_{n-1} + 1}{2n^2}. \end{aligned}$$

If we subtract $1/2$ from each side, we have the factorization

$$P_n - \frac{1}{2} = \frac{n^2-1}{n^2}\left(P_{n-1} - \frac{1}{2}\right).$$

Now, using the initial value $P_1 = 0$, we have

$$\begin{aligned} P_{2015} - \frac{1}{2} &= -\frac{1}{2} \cdot \prod_{n=2}^{2015} \frac{n^2-1}{n^2} \\ &= -\frac{1}{2} \cdot \prod_{n=2}^{2015} \frac{(n-1)(n+1)}{n \cdot n} \\ &= -\frac{1}{2} \cdot \frac{(1 \cdot 2 \cdots 2014)(3 \cdot 4 \cdots 2016)}{(2 \cdot 3 \cdots 2015)(2 \cdot 3 \cdots 2015)} \\ &= -\frac{1}{2} \cdot \frac{1 \cdot 2016}{2015 \cdot 2} \\ &= -\frac{504}{2015} \end{aligned}$$

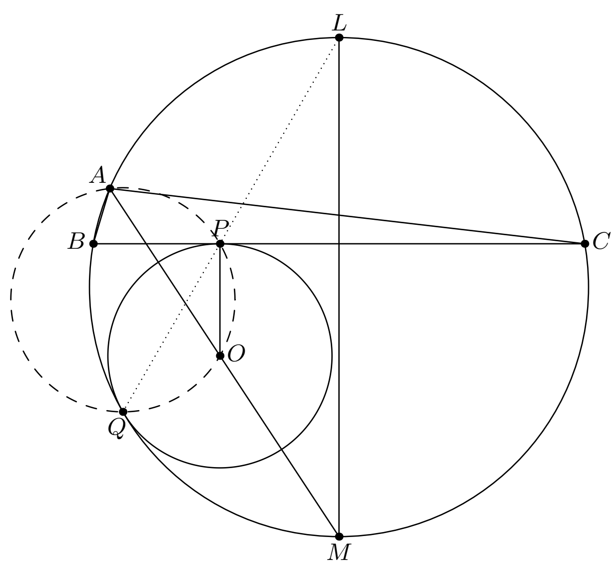
and thus

$$P_{2015} = \frac{1}{2} - \frac{504}{2015} = \frac{1007}{4030}.$$

Remark. As the number of coins tends to infinity, the corresponding probability P_n tends to $1/4$—surely a non-obvious fact?

6. Triangle ABC has circumcircle Γ. A circle with center O is tangent to BC at P and internally to Γ at Q, so that Q lies on arc BC of Γ not containing A. Prove that if $\angle BAO = \angle CAO$ then $\angle PAO = \angle QAO$.

   Solution. Let M and L be the midpoints of the arcs BC of Γ where M lies on the opposite side of line BC as A.



   We claim that the points P, Q, L are collinear. To see this, one could note that an inversion at L with radius $LB = LC$ swaps points P and Q. Alternatively, we take a homothety at Q mapping the circle with center O to Γ; since BC is a tangent, this necessarily takes Q to L.

   In any case, we can now note that OP and LM are parallel (since they are both perpendicular to BC), and by assumption points A, O, M are collinear. It follows that $APOQ$ is cyclic, as

$$\angle AQP = \angle AQL = \angle AML = \angle AOP.$$

   But $PO = QO$, so $\angle PAO = \angle QAO$.

7. Determine if there exist positive integers a, b, m, n such that $a \neq b$, $m \geq 2$, $n \geq 2$, and

$$\underbrace{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}_{m} = \underbrace{b^{b^{\cdot^{\cdot^{\cdot^{b}}}}}}_{n}.$$

Solution. There do not exist such positive integers. Assume for a contradiction that there do, however. We may assume $m > n$, so that $b > a$. Since we have equality between a power of a and a power of b, b is a rational power of a. We write $b = a^x$, where we know that $x \geq 1$ is a rational number. Then we may rewrite the b power tower (using $n \geq 2$) as

$$\underbrace{b^{b^{\cdot^{\cdot^{\cdot^{b}}}}}}_{n\ b\text{'s}} = \underbrace{a^{xa^{xb^{\cdot^{\cdot^{\cdot^{b}}}}}}}_{n-2\ b\text{'s}}$$

On the other hand, we know that this is equal to the power tower of a's in the given equation, so removing the bottom a gives

$$\underbrace{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}_{m-1\ a\text{'s}} = \underbrace{xa^{xb^{\cdot^{\cdot^{\cdot^{b}}}}}}_{n-2\ b\text{'s}}$$

In particular, this tells us that x is a rational power of a, $x = a^y$ for some nonnegative rational number y. Substituting and again removing the bottom a gives

$$\underbrace{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}_{m-2} = y + a^y \underbrace{b^{b^{\cdot^{\cdot^{\cdot^{b}}}}}}_{n-2} \tag{2}$$

Observe that the right side is y plus a rational power of a (simply a^y if $n = 2$). Now consider the possibilities for this power $a^{p/q}$. If $p/q < 0$, then $y < 0$ and the right side of (2) is less than 1, an impossibility. So either $a^{p/q}$ is irrational (again impossible) or it is the qth root of a qth power, which is necessarily a positive integer. Then y is also an integer, namely, the difference between the tower of $m - 2$ a's on the left and another power of a. Dividing both sides of (2) by y indicates that y is at least $1 - \frac{1}{a} = \frac{a-1}{a}$ times the tower of $m - 2$ a's, which is certainly larger than a power tower of $m - 3$ a's. But then a^y is larger than a power tower of $m - 2$ a's, and clearly the right side of (2) is even larger than this, so we have a contradiction.

Remark. Notice how wildly false the equality becomes. We are here using that the logarithm of a power is a product, and the logarithm of a product is a sum, so after removing just 2 a's, we get the absurd result that the *addition* of a small amount relates two power towers.