Berkeley Math Circle
Monthly Contest 4, Solutions

1. A toy slot machine accepts two kinds of coins: red and green. When a coin is inserted, the machine returns 5 coins of the other color. Laura starts with one green coin. Can it happen that after a while, she has the same number of coins of each color?

   Solution. Note that each time Laura uses the machine, the total number of coins she has increases by four, an even number. She begins with one coin, an odd number, so she will always have an odd number of coins and can never have the same number of each color.

2. Find the largest number n having the following properties:

   (a) No two digits of n are equal.

   (b) The number formed by reversing the digits of n is divisible by 8.

   Remark. n cannot start with 0, but it can end with 0.

   Solution. By condition (a), the number n cannot have more than 10 digits. Write m for the number formed by reversing the digits of n.

   The first digit of n is the last digit of m, and as such must be even, and thus at most 8. Assume that the first digit is 8.

   Then the second digit of n is the tens digit of m, and must be chosen to make the last two digits of m divisible by 4. As 98 and 78 are not divisible by 4, but 68 is, we conclude that the second digit is at most 6. Assume that the second digit is 6.

   Then the third digit must be chosen so that the last three digits of m are divisible by 8, but since 968 is divisible by 8 we derive no information from this.

   Appending the remaining digits, selecting the greatest possibility at each turn, we conclude that n is at most 8697543210. As this number indeed satisfies the conditions, it is the answer.

   Remark. Note the logical structure of this solution. Everything but the last sentence amounts to a proof that no number $n > 8697543210$ satisfies the conditions, by considering the ways that a number n can exceed 8697543210: it could have more than 10 digits, the first digit could be 9, the first two digits could be 89, 88, or 87, and so on. When all these possibilities have been eliminated, it only remains to verify that the established upper bound, 8697543210, is indeed a number satisfying the conditions (a) and (b).

3. Determine all integers n for which $n^2 + 15$ is the square of an integer.

   Remark. Because the problem asks you to "determine all integers n", you must verify that all the n you find have the desired property, and moreover prove that these are the only such integers n.

   Solution. We may limit the search to nonnegative integers n, since $n^2 + 15 = (-n)^2 + 15$. Suppose there is a nonnegative integer m such that

$$\begin{aligned} n^2 + 15 &= m^2 \\ 15 &= m^2 - n^2 \\ &= (m+n)(m-n). \end{aligned}$$

   Note that the factors $m + n$ and $m - n$ are not both negative, since their sum $2m$ is nonnegative, so they are both positive. Now the only ways to factor 15 into two positive integer factors are $15 \cdot 1$ and $5 \cdot 3$. Also $m + n \geq m - n$ since $n \geq 0$. If $m + n = 15$ and $m - n = 1$, then

$$n = \frac{(m+n) - (m-n)}{2} = \frac{15 - 1}{2} = 7$$

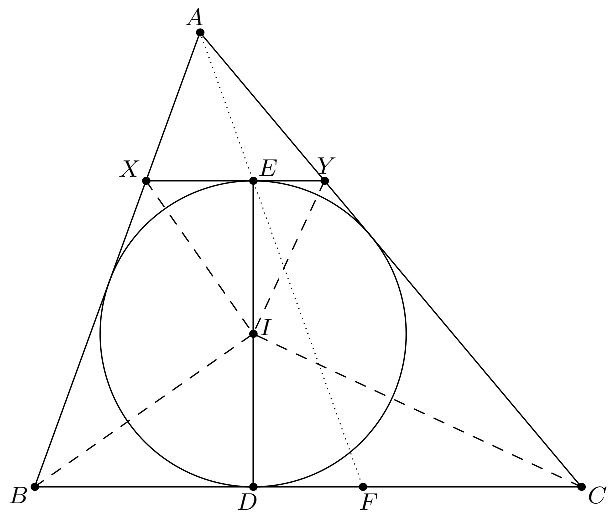
   and $m = 8$. If $m + n = 5$ and $m - n = 3$, then

$$n = \frac{(m+n) - (m-n)}{2} = \frac{5 - 3}{2} = 1$$

   and $m = 4$. Both of these solutions indeed work in the original problem. Finally, allowing for negative n, we compute the answers to be -7, -1, 1, and 7.

4. Let ABC be a triangle. The incircle, centered at I, touches side BC at D. Let E be the reflection of D through I, and let F be the reflection of D through the midpoint of BC. Prove that A, E, and F are collinear.

Solution. Let the tangent line to the incircle at E (which is of course parallel to BC) meets AB and AC at X and Y, respectively.



Note that

$$\angle IXE = \frac{\angle BXE}{2} = \frac{180 - \angle DBX}{2} = 90 - \angle DBI = BID.$$

Therefore the right triangles IXE and BID are similar. We get

$$\frac{XE}{EI} = \frac{ID}{BD},$$

that is, $XE \cdot BD = r^2$ where r is the inradius. Likewise, $EY \cdot DC = r^2$. So

$$\frac{XE}{EY} = \frac{DC}{BD} = \frac{BF}{FC}.$$

Now $XE/BF = XY/BC = AX/AB$, so $\triangle AXE \sim \triangle ABF$ and A, E, and F are collinear.

Remark. It is also possible to construct elegant solutions involving the *excircle* Ω tangent to side BC and to the extensions of rays AB and AC. In fact, Ω is tangent to BC at F (can you prove this?)

5. A *strip* of width w is the set of all points which lie on, or between, two parallel lines distance w apart. Let S be a set of n ($n \geq 3$) points on the plane such that any three different points of S can be covered by a strip of width 1. Prove that S can be covered by a strip of width 2.

Solution. Clearly if all n points are collinear we are done, so assume this is not the case. Among all $\binom{n}{3}$ of three points, consider the triangle ABC with maximal area, and suppose that BC is its longest side. Then the altitude from A to BC lies inside triangle ABC. Evidently, since ABC can be covered by a strip, we must have $h \leq 1$.

Now consider the strip of width 2 centered along line BC. For any point P outside this strip, the height from P to BC exceeds 1 so that the area of triangle PBC exceeds the area of triangle ABC. Since we chose ABC to have maximal area, it follows that all points of S are contained inside this strip, as required.

6. Show that the polynomial $(x^2+x)^{2^{1000}} + 1$ cannot be factored as the product of two nonconstant polynomials with integer coefficients.

Solution. Assume for contradiction this is not the case, and the polynomial can be written as

$$\left(x^2+x\right)^{2^{1000}} + 1 = f(x)g(x)$$

for some nonconstant f and g with integer coefficients. Clearly we may assume f and g have leading coefficient one. Taking modulo 2 we obtain that

$$f(x)g(x) \equiv (x^2+x)^{2^{1000}} + 1 \equiv \left(x^2+x+1\right)^{2^{1000}} \pmod 2$$

where the last step follows by noticing that $(a+b)^2 \equiv a^2+b^2 \pmod 2$.

As x^2+x+1 is irreducible modulo 2, it follows that we must have

$$f(x) = \left(x^2+x+1\right)^\alpha + 2\hat{f}(x)$$

and

$$g(x) = \left(x^2+x+1\right)^\beta + 2\hat{g}(x)$$

for some polynomials $\hat{f}$, $\hat{g}$ with integer coefficients, where $\alpha+\beta = 1000$. By hypothesis, $\alpha, \beta \ge 1$.

Let ε be a root to x^2+x+1, namely $\varepsilon = \frac{1}{2}\left(-1+i\sqrt{3}\right)$. Substituting ε into the original relation, we derive

$$2 = (-1)^{2^{1000}} + 1 = 4\hat{f}(\varepsilon)\hat{g}(\varepsilon)$$

and hence $\frac{1}{2} = \hat{f}(\varepsilon)\hat{g}(\varepsilon)$. But we claim this is impossible. Indeed, the right-hand side is a polynomial in ε with integer coefficients. Repeatedly applying the identity $\varepsilon^2 = -(\varepsilon+1)$, we find that the right-hand side must be of the form $a+b\varepsilon$, where a, b are integers. By equating real and imaginary parts this gives $a = \frac{1}{2}$ and $b = 0$, which is a contradiction.

7. Find all nonnegative integer solutions (a,b,c,d) to the equation

$$2^a 3^b - 5^c 7^d = 1.$$

Solution. The answer is $(1,0,0,0)$, $(3,0,0,1)$, $(1,1,1,0)$ and $(2,2,1,1)$. The solution involves several cases.

It's clear that $a \ge 1$, otherwise the left-hand side is even. The remainder of the solution involves several cases.

- First, suppose $b = 0$.
  - If $c \ge 1$, then modulo 5 we discover $2^a \equiv 1 \pmod 5$ and hence $4 \mid a$. But then modulo 3 this gives $-5^c 7^d \equiv 0$, which is a contradiction.
  - Hence assume $c = 0$. Then this becomes $2^a - 7^d = 1$. This implies $1 + 7^d \equiv 2^a \pmod{16}$, and hence $a \le 3$. Exhausting the possible values of $a = 0,1,2,3$ we discover that $(3,0,0,1)$ and $(1,0,0,0)$ are solutions.
- Henceforth suppose $b > 0$. Taking modulo 3, we discover that $5^c \equiv -1 \pmod 3$, so c must be odd and in particular not equal to zero. Then, taking modulo 5 we find that

  $$1 \equiv 2^a 3^b \equiv 2^{a-b} \pmod 5.$$

  Thus, $a \equiv b \pmod 4$. Now we again have several cases.
  - First, suppose $d = 0$. Then $2^a 3^b = 5^c + 1$. Taking modulo 4, we see that $a = 1$ is necessary, so $b \equiv 1 \pmod 4$. Clearly we have a solution $(1,1,1,0)$ here. If $b \ge 2$, however, then taking modulo 9 we obtain $5^c \equiv -1 \pmod 9$, which occurs only if $c \equiv 0 \pmod 3$. But then $5^3 + 1 = 126$ divides $5^c + 1 = 2^a 3^b$, which is impossible.
  - Now suppose $d \ne 0$ and a, b are odd. Then $6M^2 \equiv 1 \pmod 7$, where $M = 2^{\frac{a-1}{2}} 3^{\frac{b-1}{2}}$ is an integer. Hence $M^2 \equiv -1 \pmod 7$, but this is not true for any integer M.
  - Finally, suppose $b, c, d \ne 0$, and $a = 2x$, $b = 2y$ are even integers with $x \equiv y \pmod 2$, and that c is odd. Let $M = 2^x 3^y$. We obtain $(M-1)(M+1) = 5^c 7^d$. As $\gcd(M-1, M+1) \le 2$, this can only occur in two situations.
    * In one case, $M - 1 = 5^c$ and $M + 1 = 7^d$. Then $5^c + 1 \equiv 2^x 3^y$. We have already discussed this equation; it is valid only when $x = y = 1$ and $c = 1$, which gives $(a,b,c,d) = (2,2,1,1)$.
    * In the other case, $M + 1 = 5^c$ and $M - 1 = 7^d$. Taking the first relation modulo 3, we obtain that $M \equiv 2^c - 1 \equiv 1 \pmod 3$. Hence $y = 0$, and x is even. Now $2^x + 1 = 5^c$ and $2^x - 1 = 7^d$. But if x is even then $3 = 2^2 - 1 \mid 2^x - 1 \mid 7^d$, which is impossible. Hence there are no solutions here.

In summary, the only solutions are $(1,0,0,0)$, $(3,0,0,1)$, $(1,1,1,0)$ and $(2,2,1,1)$.