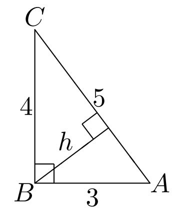
Berkeley Math Circle
Monthly Contest 3, Solutions

1. Let ABC be a triangle and suppose $AB = 3$, $BC = 4$, $CA = 5$. What is the distance from B to line AC?



   Solution. First note that since $3^2 + 4^2 = 5^2$, we have $\angle B = 90°$ by the (converse to the) Pythagorean theorem. We calculate the area of the triangle in two ways. Viewing AB as the base and BC as the height, we get that the area is $\frac{1}{2} \cdot 3 \cdot 4 = 6$. But viewing AC as the base and h (the distance to be determined) as the height, we get an equation:

$$\frac{1}{2} \cdot 5 \cdot h = 6$$
$$\frac{5}{2} \cdot h = 6$$
$$\frac{2}{5} \cdot \frac{5}{2} \cdot h = 6 \cdot \frac{2}{5}$$
$$h = \frac{6 \cdot 2}{5} = \frac{12}{5}.$$

2. Suppose a, b, c are positive integers such that

$$b = a^2 - a$$
$$c = b^2 - b$$
$$a = c^2 - c.$$

   Prove that $a = b = c = 2$.

   Remark. Note that if you try a value larger than 2 (say $a = 3$) and use the three equations to compute b, c, and then a again, the values get successively larger ($b = 6$, $c = 30$, $a = 870 \neq 3$). This is the motivation for the following proof.

   Solution. If $a = 1$, we get $b = 0$ which is impossible. So it is enough to show that a cannot be greater than 2. If $a > 2$, we have

$$b = a^2 - a = a(a-1) > a(2-1) = a.$$

   So $b > a$; in particular $b > 2$, so applying the same logic to the second equation we get $c > b$. Lastly, we have $c > 2$ so applying the same logic to the third equation we get $a > c$. We have now proved $a > c > b > a$ which is a contradiction.

3. Art and Ben play a game while sharing an $m \times n$ chocolate bar. They take turns breaking the bar into two rectangular pieces along one of the lines and eating the smaller piece. (If the two pieces are equal, they can choose which piece to eat.) Whoever is left with the last 1×1 square of chocolate loses. If Art moves first, describe all pairs (m, n) for which Ben has a winning strategy.

   Solution. The answer is all pairs (m, n) such that the ratio

$$R = \frac{m+1}{n+1}$$

   is an integer power of two, that is, the pairs $(a-1, 2^k a - 1)$ and $(2^k a - 1, a - 1)$ for $a \geq 2$ and $k \geq 0$. We will show that

   (a) If R is a power of two, then it will not remain so after one move;

   (b) If R is not a power of two, then some move will make it one.

   To prove (a), simply note that at each move, one of the dimensions is decreased at most by half, so either $m + 1$ or $n + 1$ is diminished by a factor strictly between 1 and 2. Thus the new value of R is sandwiched between two powers of two: $R/2$ and R in the first case, R and $2R$ in the second.

As for (b), assume $m \geq n$ (the other case is symmetric) so $2^k < R < 2^{k+1}$ for some $k \geq 0$. We have

$$2^k(n+1) < m+1 < 2^{k+1}(n+1).$$

These inequalities imply that one can move to a bar that is $[2^k(n+1)-1] \times (n+1)$, with R-value 2^k.

If R is initially a power of two, Ben wins by making it that way at each move; otherwise Art wins by the same strategy (in either case ending at the 1×1 square, where $R = 1$).

4. Show that there exist infinitely many triples of positive integers x, y, z which satisfy $x^{999} + y^{1000} = z^{1001}$.

   Solution. We will choose x, y, and z to be powers of 2 such that the terms x^{999} and y^{1000} are equal and z^{1001} is their sum. Writing $x = 2^a$, $y = 2^b$, $z = 2^c$, we have the conditions

$$999a = 1000b = 1001c - 1.$$

   The first equation suggests that we try $a = 1000d$, $b = 999d$, which yields the equation

$$999000d = 1001c - 1.$$

   Since $999000 = 1001 \cdot 998 + 2$, we can rewrite this as

$$2d = 1001(c - 998d) - 1.$$

   It is now clear that we can plug any odd positive integer $2k+1$ for $c - 998d$ and get an integer for d, and hence for c. The results are as follows:

$$\begin{aligned} a &= 1000(1001k + 500) \\ b &= 999(1001k + 500) \\ c &= 999000k + 499001. \end{aligned}$$

5. Suppose a, b, c are rational numbers such that

$$\begin{aligned} (a^2+1)^3 &= b+1 \\ (b^2+1)^3 &= c+1 \\ (c^2+1)^3 &= a+1. \end{aligned}$$

   Prove that $a = b = c = 0$.

   Solution. We have that $b = (a^2+1)^3 - 1$, $c = (b^2+1)^3 - 1$, and $a = (c^2+1)^3 - 1$. By direct substitution we derive that a satisfies the following polynomial equation of degree 216:

$$\left(\left(\left(\left(a^2+1\right)^3 - 1\right)^2 + 1\right)^3 - 1\right)^2 + 1\right)^3 - (a+1) = 0.$$

   We observe that the polynomial can be rewritten as

$$a^{216} + c_{215}a^{215} + \cdots + c_2 a^2 - a = 0$$

   for some integers $c_2, \ldots, c_{215}$. Hence by the Rational Root Theorem, if $a \neq 0$ then it follows that $a = \pm 1$. So $a \in \{-1, 0, 1\}$. Similarly, $b, c \in \{-1, 0, 1\}$ as well.

   But if $a = \pm 1$, then we have $b = (1+1)^3 - 1 = 7$, which is impossible. Hence only $a = 0$ can occur. Thus $a = b = c = 0$.

6. There is a stone at each vertex of a given regular 13-gon, and the color of each stone is black or white. Prove that we may exchange the position of two stones such that the coloring of all stones is symmetric with respect to some symmetric axis of the 13-gon.

Solution. First, we may assume there are more white stones than black stones, otherwise we can just reverse the roles of black and white. Let N denote the total number of black stones. We proceed by casework on the value of N.

In what follows, for any two stones A and B we define their *midpoint* to be the stone C for which $CA = CB$. Note that for a 13-gon this midpoint is uniquely determined (as 13 is odd).

Note that in what follows, if the given configuration is already symmetric then there is nothing to do.

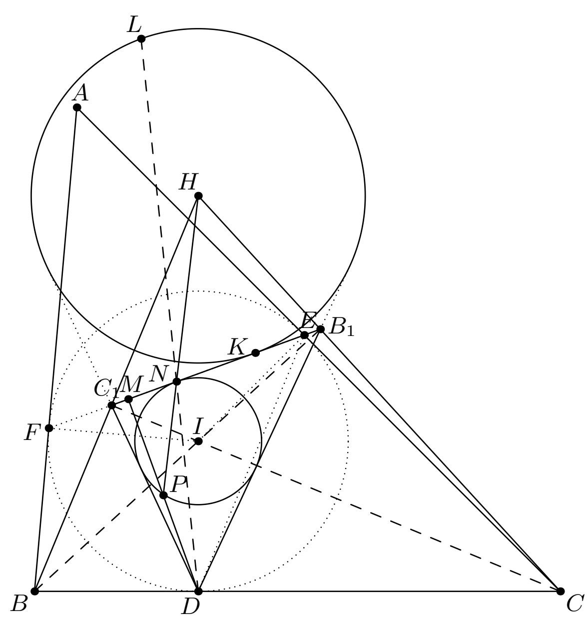
- The cases $N = 0$, $N = 1$, $N = 2$ are immediate.
- If $N = 3$, consider two black stones A and B with midpoint C. If C is already black then the configuration is already symmetric; otherwise, swap C for the third black stone.
- If $N = 4$, let A and B be two arbitrary black stones. Pick a black stone C which is not a midpoint of A and B. Then there exists a unique stone D so that $\overline{AB} \parallel \overline{CD}$. If D is black then we are done; else we swap D with the fourth black stone.
- Suppose $N = 5$. If there exists an isosceles triangle whose vertices are all black stones, then we can simply mimic the $N = 4$ case. So assume there are no isosceles triangles. Then for every $\binom{5}{2} = 10$ pairs of black stones, we consider the midpoint of the two stones, which is white by assumption. Since there are a total of 8 white stones, we find there is a white stone E which is the midpoint of two pairs (A, B) and (C, D) of black stones; hence $\overline{AB} \parallel \overline{CD}$. Then we can swap E with the fifth black stone.
- Finally, suppose $N = 6$. For every pair of $\binom{6}{2} = 15$ black stones we again consider their midpoint. Thus there is some stone X which is the midpoint of two pairs (A, B), (C, D) of black stones. Hence $\overline{AB} \parallel \overline{CD}$. Let E be a black stone other than A, B, C, D, or possibly X. Then let F denote the sixth black stone (which may be X). Then we can move F so that $\overline{AB} \parallel \overline{CD} \parallel \overline{EF}$ and this last case is complete as well.

7. Let ABC be a triangle with incenter I. The incircle of ABC is tangent to sides BC, CA, AB at D, E, F. Let H denote the orthocenter of triangle BIC, and let P denote the midpoint of the altitude from D to EF. Prove that HP bisects EF.

Solution. Without loss of generality, $AB \le AC$. Let B_1 and C_1 be the projections of C and B onto lines BI and CI, respectively. Then $\angle IEC = \angle IB_1C = 90^\circ$, so quadrilateral IEB_1C is cyclic. But then

$$\angle B_1EC = \angle B_1IC = 180^\circ - \angle BIC = 180^\circ - \angle FEC$$

and so we find that B_1 lies on line EF. Similarly, C_1 lies on line EF.



Observe that I is the orthocenter of triangle HBC. Then it is well-known that I is the incenter of triangle DB_1C_1. Moreover, H is the intersection of the external angle bisectors of $\angle DC_1B_1$ and $\angle DB_1C_1$. So it is the center of the circle Γ tangent to B_1C_1 and the extensions of rays DB_1 and DC_1 past B_1 and C_1.

Let N be the foot of the altitude from I to $\overline{B_1C_1}$, noting that N is in fact the midpoint of $\overline{EF}$. Let K be the tangency point of the Ω to $\overline{B_1C_1}$. Consider the homothety which takes the incircle of $\triangle DB_1C_1$ to the circle Γ; it sends N to the point diametrically opposite K on Ω. Call this point L; we see D, N, L are collinear by the homothety mentioned. Moreover, M, N, K are collinear. As H is the midpoint of $\overline{KL}$ and P is the midpoint of $\overline{DM}$, it follows that the points H, N, P are collinear, which is what we wanted to prove.