Berkeley Math Circle
Monthly Contest 3
Due December 2, 2014

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 1, Problem 3
Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Let ABC be a triangle and suppose $AB = 3$, $BC = 4$, $CA = 5$. What is the distance from B to line AC?

2. Suppose a, b, c are positive integers such that

$$\begin{aligned} b &= a^2 - a \\ c &= b^2 - b \\ a &= c^2 - c. \end{aligned}$$

   Prove that $a = b = c = 2$.

3. Art and Ben play a game while sharing an $m \times n$ chocolate bar. They take turns breaking the bar into two rectangular pieces along one of the lines and eating the smaller piece. (If the two pieces are equal, they can choose which piece to eat). Whoever is left with the 1×1 square of chocolate loses. If Art moves first, describe all pairs (m, n) for which Ben has a winning strategy.

4. Show that there exist infinitely many triples of positive integers x, y, z which satisfy $x^{999} + y^{1000} = z^{1001}$.

5. Suppose a, b, c are rational numbers such that

$$\begin{aligned} (a^2+1)^3 &= b+1 \\ (b^2+1)^3 &= c+1 \\ (c^2+1)^3 &= a+1. \end{aligned}$$

   Prove that $a = b = c = 0$.

6. There is a stone at each vertex of a given regular 13-gon, and the color of each stone is black or white. Prove that we may exchange the position of two stones such that the coloring of all stones is symmetric with respect to some symmetric axis of the 13-gon.

7. Let ABC be a triangle with incenter I. The incircle of ABC is tangent to sides BC, CA, AB at D, E, F. Let H denote the orthocenter of triangle BIC, and let P denote the midpoint of the altitude from D to EF. Prove that HP bisects EF.