Berkeley Math Circle
Monthly Contest 2, Solutions

1. Six consecutive prime numbers have sum p. Given that p is also prime, determine all possible values of p.

   Solution. We consider two cases.

   - If 2 is one of the six prime numbers, then the only possible sum is $p = 2+3+5+7+11+13 = 41$, which is indeed a prime.
   - If 2 is not one of the six prime numbers, then all six primes are odd, so their sum must be even. Moreover, $p > 2$. Therefore p is not prime.

   So, the only possible value of p is $p = 41$.

   Remark. In any problem which asks to "determine all", "find all', et cetera, the solution always requires two claims: first, one must show that all claimed values are achievable (here, we needed to show $p = 41$ actually occurs), and secondly, to prove that the claimed values are the only ones possible (here, we needed to show no other p could occur).

2. With two properly chosen weights and a balance scale, it is possible to determine the weight of an unknown object known to weigh an integer number of pounds from 1 to n. Find the largest possible value of n.

   Remark. The balance scale tells whether the weights placed on each side are equal and, if not, which side is heavier. It may be used an unlimited number of times.

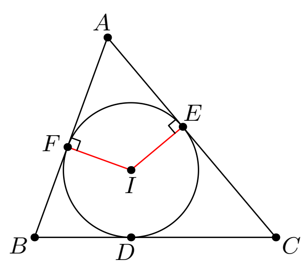
   Solution. Let a and b be the known weights. The balance scale allows one to compare the unknown weight with four known weights: a, b, $a+b$, and $a-b$ (the last of these is gotten by balancing $x+b$ on one side with a on the other). After the comparisons are done, there are at most 4 values that x can be known to equal and at most 5 gaps (3 between and 2 on the ends) for x to lie in. Thus at most 9 values of x can be distinguished.

   To show $n = 9$ is achievable, take the weights to be $a = 6$ lb and $b = 2$ lb. Then the nine possibilities can be distinguished by

   $$1 < b, \quad 2 = b, \quad b < 3 < a-b, \quad 4 = a-b, \quad a-b < 5 < a, \quad 6 = a, \quad a < 7 < a+b, \quad 8 = a+b, \quad 9 > a+b.$$

3. Let ABC be a triangle. A circle is tangent to segments BC, CA, AB at points D, E, F, respectively. Given that the measures of $\angle CAB$, $\angle ABC$, $\angle BCA$ form an arithmetic progression in some order, prove that the measures of $\angle FDE$, $\angle DEF$, $\angle EFD$ also form an arithmetic progression in some order.

   Solution. The main observation is that the three angles of a triangle form an arithmetic progression if and only if one of the angles is $60°$. Indeed, suppose that a triangle has angles $x \le y \le z$ in arithmetic progression. Then $y = \frac{x+y+z}{3} = 60°$. Conversely, if a triangle has angles x, $60°$, y, where $x \ge y$, we have $x + y = 120°$, so $x - 60° = 60° - y$.



   Suppose without loss of generality that $\angle CAB = 60°$. We will now prove that $\angle FDE = 60°$ as well. Let I be the center of the inscribed circle. Since $\angle IFA = \angle IEA = 90°$ we have that

   $$\angle EIF = 360° - 2 \cdot 90° - \angle BAC = 120°.$$

   But then

   $$\angle EDF = \frac{1}{2}\angle EIF = 60°$$

   as desired.

Remark. A second possible approach is to notice that triangles BDF and CDE are isosceles triangles, and use direct computations to show that $\angle FDB + \angle EDC = 120°$.

4. On a distant planet, there are 2014 cities, some pairs of which are connected by two-way roads. It turns out that the population of each city is the average of the populations of the cities to which it is connected by a single road, and moreover that it is possible to travel from every city to every other city by a sequence of roads.

   Prove that all cities have the same population.

   Solution. Consider the city $C_{\max}$ with the maximal population M (breaking ties arbitrarily). Then M is the average of the populations of the neighboring cities, say $p_1, p_2, \ldots, p_n$, meaning that

$$\frac{p_1 + p_2 + \cdots + p_n}{n} = M.$$

   But $p_1, p_2, \ldots, p_n \leq M$, and hence $p_1 + p_2 + \cdots + p_n \leq nM$. So this can only occur if $p_1 = p_2 = \cdots = p_n = M$. Hence all neighbors of $C_{\max}$ have population M.

   Proceeding in the same fashion, we find that all neighbors of neighbors of $C_{\max}$ also must have population M, and so on. Because the network of cities is connected, this implies that all cities must have population M.

5. Prove that, for positive integers n and m,

$$\gcd(2^m - 1, 2^n - 1) = 2^{\gcd(m,n)} - 1. \tag{1}$$

   Solution. Our main method of attacking the gcd is the following fact:

$$\gcd(m, n) = \gcd(m + kn, n)$$

   for positive integers n and k. (Proof: If $d|m$ and $d|n$, then $d|m + kn$; if $d|m + kn$ and $d|n$, then $d|m$. Thus the pairs on both sides have the same set of common divisors.)

   Suppose that (1) does not hold and $m+n$ is as small as possible (here we are using the "well-ordering principle"). Note that (1) clearly holds when $m = n$, so we may assume that $m > n$. Let $m-n = r$ and note that $\gcd(m, n) = \gcd(r, n)$ Also,

$$\begin{aligned}\gcd(2^m - 1, 2^n - 1) &= \gcd(2^m - 2^r + 2^r - 1, 2^n - 1)\\ &= \gcd(2^r(2^n - 1) + 2^r - 1, 2^n - 1)\\ &= \gcd(2^r - 1, 2^n - 1).\end{aligned}$$

   Therefore we may replace m by r without affecting the claim that (1) is false. But now $r+n < m+n$, contradicting the minimality of $m + n$.

6. Let $\mathbb{R}_{\geq 0}$ denote the set of nonnegative real numbers. Find all functions $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $x, y \in \mathbb{R}_{\geq 0}$,

$$f\left(\frac{x + f(x)}{2} + y\right) = 2x - f(x) + f(f(y)) \tag{2}$$

   and

$$(f(x) - f(y))(x - y) \geq 0. \tag{3}$$

   Solution. The only solution is $f(x) = x$. It clearly works. Note that (3) is a disguised way of saying f is increasing (i.e. $x \leq y$ implies $f(x) \leq y$).

   We claim that $f(x) \geq x$ for all x. If $f(x) < x$ for some x, let $y = \frac{x-f(x)}{2} \geq 0$ in (2). Then

$$\begin{aligned}f(x) &= 2x - f(x) + f\left(f\left(\frac{x - f(x)}{2}\right)\right)\\ &\geq 2x - f(x)\\ 2f(x) &\geq 2x\\ f(x) &\geq x,\end{aligned}$$

a contradiction. So $f(x) \geq x$.

We next find the value of $f(0) = c \geq 0$ by plugging various pairs (x, y) into (2). From $(c, 0)$ we get

$$f\left(\frac{c + f(c)}{2}\right) = 2c - f(c) + f(c) = 2c;$$

since $f(c) \geq c$ and f is increasing, we get $f(c) \leq 2c$. But from $(0, 0)$ we get

$$f(c/2) = -c + f(c);$$

since $f(c/2) \geq f(0) = c$, we get $f(c) \geq 2c$. So equality holds: $f(c) = 2c$ and $f(c/2) = c$. Finally, $(0, c/2)$ gives

$$2c = -c + 2c,$$

that is, $c = 0$.

To complete the proof, plug $y = 0$ into (2) to get

$$f\left(\frac{x + f(x)}{2}\right) = 2x - f(x),$$

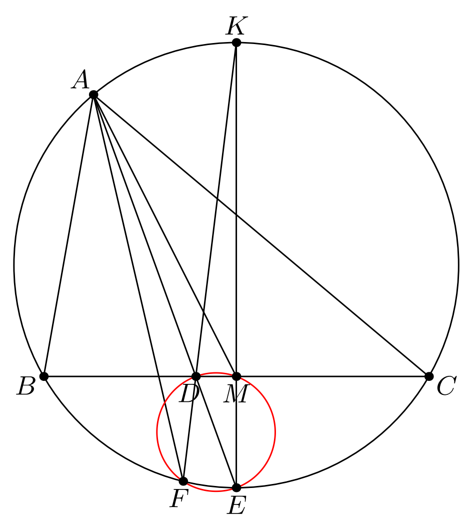
so

$$2x = f(x) + f\left(\frac{x + f(x)}{2}\right) \geq x + \frac{x + f(x)}{2} \geq x + \frac{x + x}{2} = 2x.$$

Equality holds everywhere, so $f(x) = x$.

7. Let ABC be a scalene triangle inscribed in circle Γ. The internal bisector of $\angle A$ meets $\overline{BC}$ and circle Γ at points D and E. The circle with diameter $\overline{DE}$ meets Γ at a second point F. Prove that $\frac{AB}{AC} = \frac{FB}{FC}$.

Solution. Let K be the midpoint of arc $\widehat{BC}$ containing A. Let M be the midpoint of $\overline{BC}$.



Since $\angle DME = \angle DFE = 90°$, the points D, M, E, F are concyclic. Moreover, since $\angle EFK = \angle EFD = 90°$, the points F, D, K are collinear. Finally, since $\angle KAD = \angle KMD = 90°$, the points A, D, M, K are concyclic.

Therefore,

$$\angle FAD = \angle FAE = \angle FKE = \angle DKM = \angle DAM.$$

This implies that $\angle BAF = \angle CAM$ and $\angle CAF = \angle BAM$. It follows that

$$\frac{FB}{FC} = \frac{\sin \angle BAF}{\sin \angle CAF} = \frac{\sin \angle CAM}{\sin \angle BAM} = \frac{AB \cdot \frac{BM}{\sin \angle AMB}}{AC \cdot \frac{CM}{\sin \angle AMC}} = \frac{AB}{AC}$$

as required.

Remark. The line AF is called a *symmedian* of triangle ABC, and quadrilateral $ABFC$ with $\frac{FB}{FC} = \frac{AB}{AC}$ is called *harmonic*. A second fancier approach is to use an *inversion* at A with radius $\sqrt{AB \cdot AC}$ followed by a reflection around line AD. It can be shown that this map sends F at M, implying $\angle BAF = \angle CAM$.