Berkeley Math Circle
Monthly Contest 1
Due October 1, 2014

Instructions
This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 1, Problem 3
Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

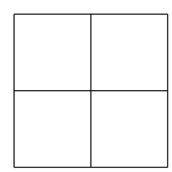
If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.
Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.
Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.
Enjoy solving these problems and good luck!

Problems

1. In the sequence
$$1, 4, 7, 10, 13, 16, 19, \ldots,$$
each term is 3 less than the next term. Find the 1000th term of the sequence.

   Remark. Simply writing down the answer will yield no credit, and appealing to a formula copied out of a math textbook will receive at most 1 point. To receive full credit, you must explain how the 1000th term is found.

2. Is it possible to draw the figure at right, without lifting your pencil from the sheet of paper,

   (a) such that each line segment is drawn only once?

   (b) such that each line segment is drawn exactly twice?

   Tricks such as folding the paper, drawing on both sides, or retracting a mechanical pencil are not permitted.

   Remark. Each part, (a) and (b), demands a separate answer of *yes* or *no*. If you think the answer is *yes*, then you must explain clearly how to perform the drawing. If you think the answer is *no*, then you must prove rigorously that it is impossible.

3. Six children are invited to a birthday party, and each pair of them are either mutual friends or mutual strangers. Prove that there are either three of them that are all friends or three of them that are all strangers to one another.

4. Determine all integers x such that the product $x(x + 1)(x + 2)$ is the square of an integer.

5. Let $ABCDEF$ be a convex hexagon such that the quadrilaterals $ABDE$ and $ACDF$ are parallelograms. Prove that $BCEF$ is also a parallelogram.

6. Determine if it is possible to color each of the rational numbers either red or blue such that the following three conditions are all satisfied:

(i) x and $-x$ are opposite colors, for all rational $x \neq 0$;

(ii) x and $1-x$ are opposite colors, for all rational $x \neq 1/2$;

(iii) x and $1/x$ are opposite colors, for all rational $x \neq 0, \pm 1$.

7. Let $a > 2$ be given, and define a sequence $a_0, a_1, a_2, \ldots$ by

$$a_0 = 1, \quad a_1 = a, \quad a_{n+1} = \left(\frac{a_n^2}{a_{n-1}^2} - 2 \right) \cdot a_n.$$

Show that for all integers $k \geq 0$, we have

$$\sum_{i=0}^{k} \frac{1}{a_i} < \frac{a + 2 - \sqrt{a^2 - 4}}{2}.$$