PROBLEM SOLVING: CIRCLES AND ANGLES IN PLANE GEOMETRY

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BMC, October 29, 2013

1. Basic Facts and Introductory Problems

Theorem 1.1. *Angle $\angle ABC$ whose vertices lie on a circle is called* inscribed angle. *If O is the center of this circle, then*

$$\angle ABC = \frac{1}{2}\angle AOC$$

if points B and O line on the same side of AC, and

$$\angle ABC = 180^\circ - \frac{1}{2}\angle AOC$$

otherwise. The most widely used corollary of this theorem is that equal chords subtend angles that are either equal or give 180° when summed.

Theorem 1.2. *The value of the angle between chord AB and the tangent to the circle that passes through point A is equal to half the angle value of arc AB.*

Theorem 1.3. *The angle values of arcs conned between parallel chords are equal.*

Problem 1.4. a) From point A lying outside of a circle, rays AB and AC come out and intersect the circle. Prove that the value of angle $\angle BAC$ is equal to half the difference of the angle measures of the arcs conned inside this angle.
b) The vertex of angle $\angle BAC$ lies inside the circle. Prove that angle $\angle BAC$ is equal to half the sum of angle measures of the arcs conned inside angle $\angle BAC$ and inside the angle symmetric to it with respect to vertex A.

Problem 1.5. From point P inside acute angle $\angle BAC$ perpendiculars PC_1 and PB_1 are dropped on lines AB and AC, respectively. Prove that $\angle C_1AP = \angle C_1B_1P$.

Problem 1.6. Prove that all the angles formed by the sides and diagonals of a regular n-gon are integer multiples of $\frac{180^\circ}{n}$.

Problem 1.7. The center of an inscribed circle of triangle ABC is symmetric to the center of the circumscribed circle w.r.t. side AB. Find the angles of triangle $\triangle ABC$.

Problem 1.8. The bisector of the exterior angle at vertex C of triangle $\triangle ABC$ intersects the circumscribed circle at point D. Prove that $|AD| = |BD|$.

2. Equal Arcs

Problem 2.1. Vertex A of an acute triangle $\triangle ABC$ is connected by a segment with the center O of the circumscribed circle. From vertex A height AH is drawn. Prove that $\angle BAH = \angle OAC$.

Problem 2.2. Two circles intersect at points M and K. Lines AB and CD are drawn through M and K, respectively. They intersect the fist circle at points A and C, the second circle at points B and D, respectively. Prove that AC is parallel to BD.

Problem 2.3. a) Given a perpendicular angle with center O and segment BC is moving such that its vertices stay on different sides of the angle. What is the locus of the midpoint of BC? b) Segment BC is extended into a right triangle $\triangle ABC$ such that points A and O lie on opposite sides of BC. Show that the locus of point A is a segment and find its length.

Problem 2.4. Diagonal AC of square $ABCD$ coincides with the hypothenuse of right triangle $\triangle ACK$, so that points B and K lie on one side of line AC. Prove that

$$BK = \frac{|AK - CK|}{\sqrt{2}} \textit{ and } DK = \frac{|AK + CK|}{\sqrt{2}}$$

Problem 2.5. Each angle of triangle $\triangle ABC$ is smaller than $120°$. Prove that inside $\triangle ABC$ there exists a point from which each side of $\triangle ABC$ is visible at $120°$. (This point is known as a Fermat Point or a Torricelli Point and has a number of other interesting properties)

Problem 2.6. A circle is divided into equal arcs by n diameters. Prove that the bases of the perpendiculars dropped from an arbitrary point M inside the circle to these diameters are vertices of a regular n-gon.

3. Angle between two chords

Problem 3.1. Points A, B, C, D in the indicated order are given on a circle. Let M be the midpoint of arc AB. Denote the intersection points of chords MC and MD with chord AB by E and K. Prove that $KECD$ is an inscribed quadrilateral.

Problem 3.2. Consider an equilateral triangle. A circle with the radius equal to the triangles height rolls along a side of the triangle. Prove that the angle measure of the arc cut off the circle by the sides of the triangle is always equal to $60°$.

Problem 3.3. Points A, B, C, D in the indicated order are given on a circle; points A_1, B_1, C_1 and D_1 are the midpoints of arcs AB, BC, CD and DA, respectively. Prove that A_1C_1 is perpendicular to B_1D_1.

Problem 3.4. Point P inside triangle $\triangle AB$C is taken so that $\angle BPC = \angle A + 60°$, $\angle APC = \angle B + 60°$, and $\angle APB = \angle C + 60°$. Lines AP, BP and CP intersect the circumscribed circle of triangle $\triangle ABC$ at points A, B, and C, respectively. Prove that triangle $\triangle ABC$ is an equilateral one.

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