Counting Regions in the Plane and More[1]

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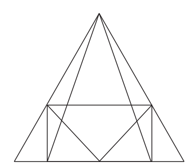
1. Overarching Problem

Problem 1 (Regions in a Circle). The n vertices of a polygons are arranged on a circle so that no three diagonals intersect in the same point. How many regions inside the circle are formed this way by all the segments connecting the n points?

Problem 2 (Preparation). Attacking Problem 1 for any n is hard. What is a good way to start on problems like this? What is your conjecture for the final answer?

2. Warm Up



Problem 2 (Warm-up). How many $\triangle$'s are in the picture?

Problem 3 (Thinking Deeper) How many *different* ways can be used to count here? Which do you prefer? Which way is "most generalizable" to other problems?

3. Special Vs. Generic Positions

Problem 5 (Understanding the Conditions). Why did Problem 1 say that no three diagonals should intersect in a point? If we allow for three or more segments to intersect in a point, will this change the answer? Will it increase it? Decrease it?

Definition 1 (General Position). Two lines in the plane are said to be in *general position* if they intersect in a point. Three (or more line) lines in the plane are said to be in *general position* if any two of them intersect, but no three (or more) lines intersect in the same point.

Problem 6 (Special Configurations). In the plane, when shall we say that two lines are in a *special position*? How about three lines in the plane in a *special position*? Draw all different configurations of three lines in the plane in a *special* position. Now draw all different configurations of three lines in the plane in a *general* position. How many special and how many generic configurations did you get?

[1]Some problems and pictures are taken from "A Decade of the Berkeley Math Circle – The American Experience," volume I, edited by Zvezdelina Stankova and Tom Rike, published by the American Mathematical Society in the MSRI Mathematical Circles Library.

Problem 7 (Pentagons and Hexagons). In a regular pentagon connect any two vertices. Are the *diagonals* in special or in generic position? How about a regular hexagon? Count in each case the number of regions into which the polygon is cut up by its diagonals.

4. Solving Predecessor Problems

Problem 1 restricted us to looking *inside* a circle? This might be why the problem is so hard! Let's look at the whole plane by temporarily eliminating the circle.

Problem 8 (Lines in Generic Position). Draw n lines in the plane so that no three intersect in the same point and no two are parallel. Into how many regions do these lines divide the plane?

Problem 9 (Games in a Tournament). n teams participated in a basketball tournament. Each team played every other team exactly once. How many games were played in total?

Problem 10 (Counting Diagonals). How many diagonals does the n-gon in our overarching problem have? What is the relation between the last three problems?

5. Poking Around and Extending the Problems

Problem 11 (Debates in a Championship). In a debate championship with n teams, each debate is done between 3 teams. How many debates can possible happen during this tournament? How about if each debate involves 4 teams?

Problem 12 (Diagonal Intersections). How many intersections of diagonals does the n-gon in our overarching problem have? What is the connection between the last two problems?

6. The Final Attack on the Overarching Problem

Problem 13 (Adding a New Segment). In the set-up of Problem 1, erase all segments, but leave the n points and the circle. Start all over the segments one at a time, in any order. Recall that the segments can be either sides or diagonals of the n-gon. Suppose a newly added segment intersects k (already drawn) diagonals. How many more regions inside the circle has this new segment added? Is there an elegant way to phrase the answer without mentioning the number k?

Problem 14 (Putting All Together). Use any of our results so far to calculate the total number of regions inside the circle. Is there an elegant way to phrase the answer? Did you check it for correctness for $n = 1, 2, \ldots, 6$? What does your answer predict for $n = 7$? Check it by brute force, if you don't believe it.

7. Optional Homework for the Die-Hards

Problem 15 (Summing Up). Prove the following formulas for any $n \geq 1$:

(a) $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

(b) $1 + 3 + 5 + 7 + \cdots + (2n-1) = n^2$.

(c) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Problem 16 (Off to Space). Generalize the overarching problem to 3 dimensions. The n vertices of a polyhedron are arranged on a sphere so that when among the planes connecting three of the points, no three intersect in the same line. How many regions inside the sphere are formed this way by all these planes?

Problem 17 (Polygons). Count the number of regions inside a (convex) n-gon made by connecting any two of its vertices if:

(a) no three diagonals intersect at the same point;
(b) the n-gon is *regular*.

Problem 18 (Binomial Theorem). The number of ways to choose k objects out of n objects is denoted by the *binomial coefficient* $\binom{n}{k}$ (read "n choose k"). Prove that this binomial coefficient is calculated by the formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ \textit{for any} } n, k \geq 0,$$

where 0! is defined to be 1 in order to make the formula work when n or k is 0. When $n < k$ or one of the n or k is < 0, then the binomial coefficient $\binom{n}{k} = 0$.

Problem 19 (Ultimate Summations). Prove the following formulas:

(a) $\dbinom{n}{0} + \dbinom{n}{1} + \dbinom{n}{2} + \dbinom{n}{3} + \cdots + \dbinom{n}{n-1} + \dbinom{n}{n} = 2^n$ for all $n \geq 0$.

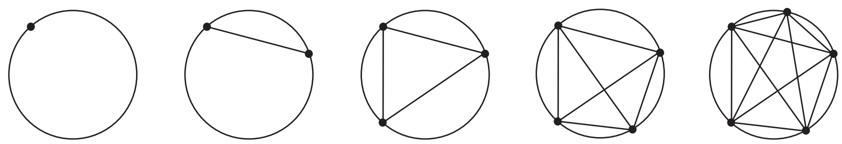
(b) $\dbinom{n}{0} + \dbinom{n}{2} + \dbinom{n}{4} + \dbinom{n}{6} + \cdots + \dbinom{n}{n-1} = 2^{n-1}$ for n odd.[2]

Problem 20 (Coincidence?) The answer 31 in the case of $n = 6$ points in the overarching Problem 1 broke the "pattern" of powers 2^k and made us rethink what is going on. The answer to the warm-up Problem 2 was also 31. Is this a coincidence, or is there a way to transform one problem into the other in this case?

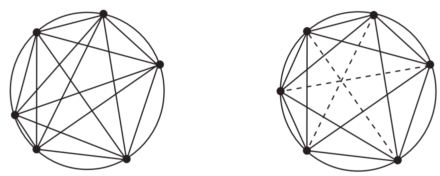
[2]For n even, the analogous sum $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \cdots + \binom{n}{n-2} + \binom{n}{n}$ is a lot harder to calculate, but it turns out that parts of this particular sum is connected to our overarching problem.

8. Short Answers and Ideas to Selected Problems

Problem 1. Wait until the end! No hurry. Keep thinking.



Problem 2. Start with small n's. For $n = 1, 2, 3, 4, 5$ we get $1, 2, 4, 8, 16$ regions in the circle. A pattern becomes apparent: *powers of* 2, i.e., $2^0, 2^1, 2^2, 2^3$, and 2^4. Thus, for $n = 6$ the number of regions must be $2^5 = 32$.... Yet, is it?



Problem 3. Start with the biggest triangle and keep adding segments, counting along the way and trying to use the symmetry of the picture as much as possible:

$$1 + 2 + 3 + 1 + 1 + 4 + 2 + 2 + 3 + 3 + 4 + 5 = 31. \quad \square$$

Problem 4. There are at least *five* possible ways of counting the triangles.

(a) Brute-force (no system): Count randomly and hope that you didn't miss and you didn't overcount anything.

(b) Incrementally: Reconstruct the picture from scratch, adding incrementally simple parts (such as segments or small triangles) and counting how many more figures of the wanted type are added at each step.

- Start with the largest triangle, add up a segment at a time and count how many *new* triangles have been created with that segment as their side or part of their side.
- Start with a "basic" triangle, i.e., a triangle which is not subdivided by another segment into smaller triangles; add up a triangle at a time and count how many *new* triangles have been created that contain your added triangle as part of them.

(d) By number of parts: Count first the triangles that are made of one piece, then those made of two pieces, then of three pieces, and so on.

(e) Symmetry: Use the central vertical line as a symmetry "divider"; count triangles on one side of it and multiply by 2, then count the triangles that go across the symmetry line, and then add. Did you get 31?

Problem 5. The number of regions will decrease if we allow for three or more diagonals to intersect in a point. For example, when $n = 6$ (see above), the number of regions is 30, as the small central triangle has degenerated to the point of intersection of three diagonals.

Problem 6. Two lines in the plane are in a special position if they are parallel (no points of intersection). There are 3 *special* configurations of three lines in the plane: the three lines intersect in the same point; or two lines are parallel and the third line in a transversal (intersects the other two lines); or all three lines are parallel.

There is only 1 *generic* configuration of three lines in the plane: a triangle whose sides have been extended to our three lines.

Problem 7. The diagonals in a regular pentagon are in a generic position, but the diagonals in a regular hexagon are in a special position: three of them intersect in a point. We have done, in some sense, the counting of the regions inside a pentagon and inside a regular hexagon already: subtract the rounded regions made by the circle in the overarching problem discussion above. For a regular pentagon, you will get $2^4 - 5 = 11$, and for a regular hexagon: $30 - 6 = 24$.

Problem 8. Start with 0 lines dividing the plane into 1 region. If you already have drawn k lines in general position, adding a $(k+1)$st line increases the number of regions by $k+1$ (why?). Hence, overall for n lines we get $1 + (1 + 2 + 3 + \cdots + n)$ regions in the plane.

Problem 9. Let's first solve the problem for *five* teams.

PST 1. Arrange the teams in one line and count the number of games, making sure that you are <u>not overcounting</u> any games.

For 5 teams, this calculation boils down to $4 + 3 + 2 + 1 = 10$.

When the problem is generalized to n teams, the result is $1 + 2 + 3 + \cdots + (n-1)$, which requires a different argument to be calculated in a concise form.

PST 2. Count the number of games played by each team *by double-counting* them – once for one of the teams and another time for the other team, and then divide by 2.

For 5 teams, this yields again

$$\frac{\big(5 \text{ teams}\big) \cdot \big(4 \text{ games each}\big)}{2} = 10.$$

Fortunately, this PST works for any n teams too:

$$\frac{\big(n \text{ teams}\big) \cdot \big((n-1) \text{ games each}\big)}{2} = \frac{n(n-1)}{2}.$$

As a result, we proved the following:

Theorem 1. The sum of the first n natural numbers is given by the formula

$$1 + 2 + 3 + \cdots + n = \frac{n(n-1)}{2}.$$

Both sides of this formula counted differently the same quantity: the <u>number of games</u> to be played among n teams in a tournament.

PST 3. To prove a formula LHS = RHS, find a quantity that equals both sides and calculate it in two different ways.

In another attempt to prove the above formula, draw an $n \times n$ table to record the scores of all games. Along the diagonal write X's, since no team plays against itself. We are not interested in the outcomes of the games; so what is written otherwise in the table does not matter. What matters are the number of non-diagonal cells: for every cell under the diagonal there is a cell above the diagonal that refers to the same game, e.g., a cell for the game between teams i and j and a cell for the game between teams j and i. Thus, the number of games is the half of the non-diagonal cells:

$$\frac{n^2 - n}{2} = \frac{n(n-1)}{2},$$

where n^2 are the cells of the whole table, the subtracted n are the diagonal cells, and the division by 2 offsets the double-counting of the games. □

PST 4. To calculate a positive integer quantity, find a table whose cells approximate this quantity. Start <u>overcounting</u> the quantity by the number of cells in the table, and then subtract and divide as necessary to eliminate unwanted cells or cells that have been counted multiple times.

In a final approach to solve the problem, we realize that the number of games corresponds to the number of pairs of teams that we can select from the given n teams. There is a quantity in combinatorics which counts even more general objects: these are the binomial coefficients $\binom{n}{k}$ from Problem 18. Applying the Binomial Theorem for $k = 2$ yields again

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{1 \cdot 2 \cdots (n-2) \cdot (n-1) \cdot n}{(1 \cdot 2)(1 \cdot 2 \cdots (n-2))} = \frac{(n-1)n}{2}.$$

Problem 10. A diagonal runs between any two vertices of polygon, except we have to subtract the n sides of the polygon since they count as diagonals. From above, there are $\binom{n}{2}$ pairs of vertices, so the answer is

$$\binom{n}{2} - n = \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}.$$

Back to Problem 8, we can now give a nice formula for the number of regions in the plane made by n lines in generic position:

$$1 + (1 + 2 + \cdots + n) = 1 + \frac{n(n-1)}{2} = 1 + \binom{n}{2}.$$

Problem 11. $\displaystyle\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$ and $\displaystyle\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{24}$.

Problem 12. Any two intersecting diagonals form a (convex) quadrilateral, and conversely, any four points on the circle form a (convex) quadrilateral whose diagonals intersect in a point. Thus, the number of diagonal intersections equals the number of 4-tuples of points, i.e., $\binom{n}{4}$.

Problem 13. The new segment will increase the number of regions by $k+1$ (why?). This works even if the segment is a side of the polygon, i.e., it has no diagonal intersections – it will still add $0 + 1 = 1$ new regions. To record this elegantly, we will say that every diagonal intersection is responsible for adding 1 new region, and every segment is responsible for adding 1 new region too.

Problem 14. From Problem 13, the number of regions equals 1 more than the number of diagonal intersections plus the number of segments (diagonals and sides of the polygon, all counted). Problems 12 and 10 give the answers to the required two quantities: $\binom{n}{4}$ and $\binom{n}{2}$. The "1 more" comes from the fact that for 1 point on the circle we start with 1 region (the whole circle), and then every additional segment adds the amount of regions we discussed in Problem 13.

Overall, the number of regions inside the circle is:

$$1 + \binom{n}{2} + \binom{n}{4} = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} \text{ for all } n \geq 1.$$

For $n = 1, 2, 3, 4, 5$ this sum does equal to the corresponding power of 2, 2^{n-1}, but for $n = 6$ and 7 we get

$$\binom{6}{0} + \binom{6}{2} + \binom{6}{4} = 1 + 15 + 15 = 31, \text{ and } \binom{7}{0} + \binom{7}{2} + \binom{7}{4} = 1 + 21 + 35 = 57.$$

If you are familiar with functions and, specifically, *polynomials*, the answer can be expressed as a degree 4 polynomial in n:

$$f(n) = 1 + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)(n-3)}{24} = \frac{1}{24}n^4 + an^3 + bn^2 + cn + 1,$$

where we leave the interested reader to calculate the coefficients a, b, and c. $\square$

Problem 15-20. You are on your own. Only, don't attempt these alone at home: ask someone for help and view them as a long-term project. :)