To infinity and beyond:
The Theory of Sets
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What is a counting number?

A *set* is any collection of objects of any nature. The "objects" are called *elements* of this set. A set A is considered given, if for every object in the Universe it is known whether it is an element of set A or not.

`Definition:` Two sets of objects are said to *have the same number of elements*, if their elemets can be matched with each other. (*Source:* A kindergarten math book.)

`Examples:` In a full classroom, there are as many chairs as there are students (because each chair is occupied by a student). There are as many working days of the week as there are fingers on my right hand (because the days can be *counted* by unbending the fingers: Mo-Tu-We-Th-Fr).

An abstract question: What is a number?

An equally abstract answer: A counting number, for example 5, is the class of all those sets whose elements can be matched with fingers on my right hand. By saying that a set has 5 elements we mean that the set belongs to this class; that is, that the elements of this set can be matched with fingers of my right hand.

More generally, "To have the same number of elements" is an *equivalence relation* between sets. Namely, all sets are partitioned into classes according to whether they have the same number of elemets or not. All sets which have the same number of elements (that is, whose elements can be matched with each other) form one equivalence class. Two sets whose elements are impossible to match (for instance, all days of the week, and all months of the year) belong to different equivalence classes. The point is: all this makes sense for infinite sets as well.

Hilbert's Hotel

It is a 5-star hotel situated on inter-star track I-5 at the outscirts of the Milky Way galaxy.

One day, a visitor knocks on the door:
"Do you have a vacant room?"
"All rooms are occupied," says the guy at the reception desk.
"Could you find a room for me, please?" begs the guest, "I've traveled for 5 light years and need rest."
"No problem, here is your key from room number 1," is the reply.
Question: How can this be?

Answer: Hilberts' Hotel has infinitely many rooms numbered 1, 2, 3, ... All rooms are occupied, but when the new visitor is given the key for room 1, the occupant of room 1 moves to room 2, of room 2 to room 3, and so on. *Conclusion:* $1 + \infty = \infty$.

`Problem 1.` Invent stories about Hilbert's Hotel which show that

$$5 + \infty = \infty, \quad \infty + \infty = \infty, \quad \infty \times \infty = \infty.$$

`Problem 2.` Let $\mathbb{Z}_+$ denote the set of all positive integers $1, 2, 3, ...$, and $\mathbb{Q}_+$ the set of all positive fractions m/n. Do these sets have the same number of elements?

Continuum

`Problem 3.` Which of the following sets of points in the plane have the same number of elements, and which do not?

(a) closed interval $[A, B]$ (including the endpoints)
(b) semi-interval $[A, B)$ (the endpoint B is excluded)
(c) open interval (A, B) (both ends are excluded)
(d) a straight line
(e) s circle (the curve)
(f) a disk (the region enclosed by the circle)
(g) a square
(h) a triangle

The tribes of Mumbo and Jumbo

`Problem 4.` These tribes argue whose language is richer. The Mumbo people claim that their language is richer, because there exists a Jumbo–Mumbo dictionary where all words of the Jumbo language are translated by distinct words of the Mumbo language, while some words of the Mumbo language remain unused. The Jumbo people claim that their language is richer, because there exists a Mumbo–Jumbo dictionary, where all words of Mumbo are translated by distinct words of Jumbo, while some words of the Jumbo language remain unused. Prove that Mumbo and Jumbo have the same number of words; that is, construct yet another dictionary which matches exacty all words of one language to all words of the other.

`Remark:` This is known set theory as *The Cantor–Bernstein Theorem.* It says that if set A has the same number of elemens as some subset of set B, and B has the same number of elements as some subset of A, then A and B have the same number of elemnts.

`Exercise:` Use the theorem to prove that (a–e) have the same number of elements, and (f–h) have the same number of elements.

Problem 5. Prove that an interval $[0,1]$ of the number line has the same number of elements as the unit square: $[0,1]\times[0,1]$.

Hint: Let $x = .x_1x_2x_3...$ and $y = .y_1y_2y_3...$ be coordinated of a point in the square writtent in their decimal representation. Take the corresponding point $z = .z_1z_2z_3...$ on the segment $[0,1]$ to be $.x_1y_1x_2y_2....$

Cantor's diagonal argument

Definition. A set which has the same number of elements as $\mathbb{Z}_+ = \{1,2,...\}$ is called *countable*. A set that has the same number of elements as an interval (say $[0,1)$) is called *continuum*.

Theorem. *Continuum is uncountable.*

Proof. We'll prove that any countable list of points of $[0,1)$ does not contain all such points. Let this be the list:

$$\begin{array}{ll} 1 & .x_1x_2x_3x_4... \\ 2 & .y_1y_2y_3y_4... \\ 3 & .z_1z_2z_3z_4... \\ 4 & \cdots \\ ... & \end{array}$$

where the rows are decimal representations of the point $x, y, z, \ldots$ of the list, and the left column is their numbers in the list. (Let's assume for the sake of uniqueness of decimal representation, that a tail of 9s is not allowed.) Following Cantor, consider the sequence of the digits on the diagonal of the table: $.x_1y_2z_3\ldots.$ Then construct another sequence, $.\bar{x}_1\bar{y}_2\bar{z}_3\ldots$, replacing every occurence of digit 0 with 1, and digits other than 0 with 0. For example, if the diagonal sequence is .31029109909..., then the newly constructed sequence is .00100010010.... Then the point in $[0,1)$ reprezented by the decimal fraction

$$.\bar{x}_1\bar{y}_2\bar{z}_3\ldots$$

is not on the list. Indeed, it differs from the 1st row by the 1st digit, from the 2nd row by the 2nd digit, from the 3rd row by the 3rd digit, and so on.