The Pigeonhole Principle

Michael Larsen

March 18, 2014

If you put $n + 1$ objects into n boxes, you are bound to end up with at least two in the same box. That's all there is to the *pigeonhole principle* (which you may sometimes also hear called *Dirichlet's principle*). There are some variants: if you put $mn + 1$ objects into n boxes, you will end up with at least $m + 1$ objects in one box; if you put infinitely many objects into a finite number of boxes, at least one box will have an infinite number of objects.

It's hard to believe such a simple idea could have useful applications, but in fact it comes up surprisingly often. The first difficulty is recognizing a pigeonhole problem when one comes along. If a problem calls for two elements of a set to have some property in common or to be near one another in some way, that is a good indication. Once you have decided the pigeonhole principle applies, a second difficulty is figuring out what the "boxes" should be. Often this is a question of grouping possible choices in a suitable way. Sometimes, when you have done this, you are finished, but another trick may be needed to take advantage of the fact that two objects land in the same box.

Here are some problems which illustrate the variety of possible applications of the pigeonhole principle. Some are relatively straightforward, but some are very hard even if you know that they are pigeonhole problems.

1. Five points are chosen in a unit square. Prove that some pair of them are within distance $1/\sqrt{2}$ of each other.

2. A room contains $n \geq 2$ people, some of whom shake hands. Show that there are two people who each end up shaking hands with the same number of other people.

3. A hexagon is inscribed in a unit circle. Prove that the shortest side has length ≤ 1.

4. Show that if n is odd, there exists $m > 0$ such that 2^m is congruent to $1 \pmod{n}$.

5. Show that the decimal representation of some power of 2 begins either 999 or 1000.

6. Show that there exists some n, $0 < n < 1,000,000$, such that the nth Fibonacci number is divisible by $1,000$. Recall that $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$.

7. Given 9 points $P_0, P_1, \ldots, P_8$ in $\mathbf{R}^3$, show that there exist distinct positive integers i and j such that $\angle P_i P_0 P_j$ is at most $\pi/2$.

8. Show that every $n+1$ element subset of $\{1, 2, 3, \ldots, 2n\}$ contains two distinct elements one of which divides the other.

9. Show that there exists an integer which can be written as a sum of 3 perfect squares in at least 2014 different ways.

10. Show that for each n there exist finitely many n-tuples of positive integers $(a_1, \ldots, a_n)$ such that $1/a_1 + \cdots + 1/a_n = 1$.

11. Show that given five real numbers $x_1 < x_2 < \ldots < x_5$ there exist i and j such that $0 < \frac{x_i - x_j}{1 + x_i x_j} < 1$.

12. Show that if p is a prime and $n^2 + 1 \equiv 0 \pmod{p}$, then p can be written as a sum of two squares.

13. Show that for every positive integer N, there exist integers m, n such that $0 < n \leq N$ and $|m - n\sqrt{2}| < 1/N$. Deduce that there are infinitely many solutions in integers m and n to $|m^2 - 2n^2| = 1$.

14. Show that embedded in any sequence of 101 numbers $x_0, x_1, \ldots, x_{100}$, all different, is a subsequence of 11 terms (not necessarily consecutive) which is either an increasing sequence or a decreasing sequence.