

Berkeley Math Circle
RANDOM WALKS AND GAMBLING

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- 1 A deck of cards is randomly cut into three piles. I bet that at least one of the cards on the top of a pile is a “face card;” i.e., a Jack, Queen, or King? Do you want to take this bet?
- 2 Two players alternately toss a penny, and the one that first tosses heads wins. What is the probability that
 - (a) the game never ends?
 - (b) the first player wins?
 - (c) the second player wins?
- 3
 - (a) On average, how many times must a die be thrown until one gets a 6?
 - (b) How many times, on average, should one toss a fair die in order to see all 6 possible outcomes?
- 4 We each have a shuffled deck of cards and we deal our cards one at a time and compare. I bet you that we will have at least one match (for example, our 7th cards are the same). Do you want to take this bet?
- 5 *The Classic Birthday Problem.* In a room of 30 people, how likely is it that at least two people have the same birthday?
- 6 *Another Birthday Problem.* In Klopstockia, all factory workers must work every day of the year, getting no days off, with only one exception: whenever it is a worker’s birthday, the factory closes down and everyone gets the day off! Factories are obligated to hire workers randomly, without discriminating on the basis of birthday. In order to maximize output, how many workers should a factory employ?
- 7 It costs a consumer \$1 to buy a Klopstockia lottery ticket. The buyer then scratches the ticket to see the prize. Compute, to the nearest penny, the expected profit that the state of Klopstockia makes per ticket sold, given the following scenarios for prizes awarded. (The state will make a profit if the expected value of the lottery ticket is *less* than \$1.)

(a)

Prize	\$1	\$10
Probability	$\frac{1}{10}$	$\frac{1}{1,000}$

(b)

Prize	\$1	\$10	a free lottery ticket
Probability	$\frac{1}{10}$	$\frac{1}{1,000}$	$\frac{1}{5}$

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 - (a) the game never ends?
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9 *The Classic Gambler's Ruin Problem.* Two players take turns tossing a fair coin. If the coin is heads, player A gives player B a dollar. If the coin turns up tails, B gives A a dollar. Player A starts with a dollars, and player B starts with b dollars (a and b are non-negative integers). Once a player goes bankrupt (i.e., has zero dollars) the game is over. What is the probability that A goes bankrupt?

What happens if the probabilities are not equal; i.e., what if the probability that the coin is heads is p , for some fixed $0 \leq p \leq 1$.

10 *A Bug on a Cube.* Imagine a bug that crawls along the edges of a cube. The bug does not change directions while traveling on an edge. Two adjacent vertices, F and P , have food and poison, respectively. If the bug reaches either of these vertices, it stops traveling. Whenever the bug reaches one of the other six vertices, it has a choice of three edges on which to travel and it chooses randomly (i.e., with probability of $1/3$ for each choice). For each of these six starting vertices, compute the probability that the bug lives (i.e., reaches F before reaching P).

11 *What a Loser!* You arrive in Las Vegas with \$100 and decide to play roulette, making the same bet each time, until you are either bankrupt or have doubled your money. Which of the following strategies is best?

- (a) Making bets of \$1 each time.
- (b) Making bets of \$10 each time.
- (c) Making a single bet of \$100.

12 *A Gambling "System."* Suppose you are playing a game with a 50% chance of winning each time. You can bet any amount, and if you win, you win twice your bet. If you lose, you lose your bet. In other words, if you bet B dollars, your *profit* is $\pm B$ depending on whether you win or lose. You decide that you will play, stopping as soon as you win, doubling the size of your bet each time. You are guaranteed to make a profit? Right? Use expectation to show that this won't work. What if you triple instead of double?

13 *The St. Petersburg Paradox.* Consider the following game. I will flip a fair coin until it shows up heads. We keep track of the number of flips until this happens. If it happens on the first flip, I'll pay you \$2. If it takes two flips, then I'll pay you \$4. Three flips, \$8, etc. In other words, if it takes n flips until the first head, I will pay you 2^n dollars. Pretty sweet game!

How much is this game worth *to you*? In other words, if there was a ticket that allowed you to play the game once with me (I flip the coin until it is heads, and pay you the appropriate amount), how much would you pay for the ticket? Clearly, you'd pay at least 1 dollar. In fact, you'd almost certainly pay 2 dollars. How about 3? 4? 5? More?

14 *A Problem from The 2000 Bay Area Math Meet (BAMM).* Consider the following experiment:

- First a random number p between 0 and 1 is chosen by spinning an arrow around a dial which is marked from 0 to 1. (This way, the random number is "uniformly distributed"—the chance that p lies in the interval, say, from 0.45 to 0.46 is exactly $1/100$; and the chance that p lies in the interval from 0.324 to 0.335 is exactly $11/1000$, etc.)
- Then an unfair coin is built so that it lands "heads up" with probability p .
- This coin is then flipped 2000 times, and the number of heads seen is recorded.

What is the probability that exactly 1000 heads were recorded?

15 *A Problem from BAMM 1999.* Eleven points are chosen randomly on the surface of a sphere. What is the probability that all eleven points lie on some hemisphere of this sphere?