Bulgarian Math Olympiads: the Ultimate Challenge, Part III

by Zvezdelina Stankova

Berkeley Math Circle – Beginners Group

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During our second session on September 10, 2013 we solved (much!) harder versions of two of our original problems.

1. A Pre-Sudoku Puzzle

Problem 1. A 4×4 square table is filled with the numbers $1, 2, 3$, and 4 in such a way that every number appears once in every row, column, and diagonal. What number is A?

1	2	3	4
A			
	1	4	3

(1B) **(Ultimate version)** What is the *minimal* initial set of numbers (anywhere in the table) that will allow you to fill in the table in a *unique* way? Why?

Answer to (1B): The minimal set of initial information to force the problem to have a unique solution consists of 3 numbers.

Proof: The first part of the proof consists of a *specific example* with 3 numbers that are sufficient to full in the table *uniquely*. Check that, indeed, the only way to fill in the table is the second table on the right. There are many other triplets of numbers which will cause the puzzle to be solved uniquely.

1	2		
			3

$\longrightarrow$

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

The second part of the proof is a *logical argument* that explains why no two given numbers (anywhere in the table) are sufficient for a unique solution. Indeed, say, two different numbers a and b are placed anywhere in the table. If one can fill the table one way, then switch in this solution the numbers c and d (the other two numbers that were not initially placed in the table) to obtain yet another, *different* filling of the table! Hence, if there is one solution, there will be at least one more solution. As an example, if you start with 2 and 3 in the position below, you will be able to fill the table in the original way as above; but you can switch then 1 and 4 to obtain a second solution to the problem:

	2		
			3

$\longrightarrow$

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

OR (switching $1 \leftrightarrow 4$):

4	2	3	1
3	1	4	2
1	3	2	4
2	4	1	3

It is not surprising that the problem now can be generalized to any $n \times n$ table to be filled with the numbers $\{1, 2, 3, \ldots, n\}$ according to the same rule: every row, table and diagonal must have different numbers in it. In fact, our discussion is very much related to the famous Sudoku 9×9 puzzles with the additional condition that every of the nine 3×3 sub-tables also contain exactly once the numbers from 1 to 9 (cf. Tom Davis's article at http://math.stanford.edu/circle/notes08f/sudoku.pdf).

2. Conjecturing, Inducting, and Telescoping

Problem 2. What is the sum $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\dfrac{1}{4\cdot 5}+\dfrac{1}{5\cdot 6}$ equal to?

(2B) **(Ultimate versions)** What are the following sums equal to:

- $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\dfrac{1}{4\cdot 5}+\dfrac{1}{5\cdot 6}+\cdots+\dfrac{1}{2013\cdot 2014}=\,?$
- $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\dfrac{1}{4\cdot 5}+\dfrac{1}{5\cdot 6}+\cdots+\dfrac{1}{n(n+1)}=\,?$
- $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\dfrac{1}{4\cdot 5}+\dfrac{1}{5\cdot 6}+\cdots+\dfrac{1}{n(n+1)}+\cdots=\,?$

First solution. In solving the original problem, one can apply a brute-force approach. Using that the least common denominator of the 6 fractions is 60, we can modify all fractions to have that denominator:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}=\frac{30}{60}+\frac{10}{60}+\frac{5}{60}+\frac{3}{60}+\frac{2}{60}=\frac{30+10+5+3+2}{60}=\frac{50}{60}=\frac{5}{6}\cdot \qquad \square$$

Unfortunately, this brute-force approach will become very cumbersome if we try to solve the second version of the problem, where the fractions run until $\frac{1}{2013\cdot 2014}$, not to talk about the third version that does not specify until which the fraction the sum runs, or ... the ultimate fourth version of the problem where we add *all*, infinitely many, such possible fractions $\frac{1}{n(n+1)}$! A new method is necessary.

Second solution. The answer of $5/6$ to a problem that ends in the fraction $1/(5\cdot 6)$ is highly suspicious. So, we calculate a few more initial cases to see if there is a pattern. Indeed, check by brute-force or other ways that:

$$\frac{1}{1\cdot 2}=\frac{1}{2};\ \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{2}{3};\ \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\frac{3}{4};\ \text{and}\ \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}=\frac{4}{5}\cdot$$

To make this precise and formal, we invoke the **Method of Mathematical Induction (MMI)**. We will show that the following statements S_n are all true:

Conjecture 1. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\dfrac{1}{4\cdot 5}+\dfrac{1}{5\cdot 6}+\cdots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$ for all $n \geq 1$.

<u>(1) Base Case:</u> For $n=1$, S_1 reads $\frac{1}{1\cdot 2}=\frac{1}{2}$, which is certainly true.

(2) Inductive Step: We will assume that S_n is true for some $n \geq 1$, and prove that S_{n+1} is also true. Let's start from the left-hand side (LHS) of S_{n+1}:

$$\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\cdots+\frac{1}{n(n+1)}\right)+\frac{1}{(n+1)(n+2)}.$$

The expression in the parentheses is precisely the LHS of S_n, which we know how to calculate: it equals $1/n(n+1)$ by our assumption (*inductive hypothesis*). Thus, the LHS of S_{n+1} equals:

$$\frac{n}{n+1}+\frac{1}{(n+1)(n+2)}=\frac{n(n+2)}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)}=\frac{n(n+2)+1}{(n+1)(n+2)}=\frac{n^2+2n+1}{(n+1)(n+2)}.$$

Recall that we ultimately want to get the answer $(n+1)/(n+2)$, i.e., there should be no $(n+1)$ in the denominator of our fraction; hence we must be able to cancel $(n+1)$ from both denominator and numerator of our fraction. Looking closely at the numerator: $n^2+2n+1=(n+1)^2$ (check it!), so that our answer becomes:

$$\frac{n^2+2n+1}{(n+1)(n+2)}=\frac{(n+1)^2}{(n+1)(n+2)}=\frac{n+1}{n+2},$$

which is precisely the desired right-hand side (RHS) of our statement S_{n+1}.

(3) Conclusion: By MMI, we can conclude that *all* statements S_n are true for $n=1,2,3,\ldots$, i.e., the pattern we discovered persists forever:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1},\ \text{for every } n\geq 1. \quad \square$$

The method of induction is hard to grasp from the first time you see it and it is even harder to apply it to problems from the get-go, However, once you get used to it, it turns into one of the most powerful "weapons" of mathematical discovery you can have in your arsenal, because it allows you to prove *infinitely many* statements (equalities, inequalities, and what-not) by doing only *finitely many* steps.

Third solution. There is a, perhaps, even more astonishing way of proving our concise formula for the sum. In Calculus, it is usually referred to as the *telescoping method* for series. It relies on the following simple observation:

$$\frac{1}{1\cdot 2}=\frac{1}{1}-\frac{1}{2},\ \frac{1}{2\cdot 3}=\frac{1}{2}-\frac{1}{3},\ \frac{1}{3\cdot 4}=\frac{1}{3}-\frac{1}{4},\ \ldots.$$

It is good to check all of these by hand individually, before verifying with us the general case for any n, using some algebra:

$$\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1},\ \text{for } k\geq 1.$$

Indeed, to prove this, start from its RHS and modify it to make it equal to the LHS:

$$\frac{1}{k}-\frac{1}{k+1}=\frac{1\cdot(k+1)}{k(k+1)}-\frac{1\cdot k}{(k+1)k}=\frac{(k+1)-k}{k(k+1)}=\frac{1}{k(k+1)}.$$

Substituting our new differences for each of our original fractions, we rewrite our sum:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\cdots+\frac{1}{n(n+1)}=$$

$$=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)=$$

$$=\frac{1}{1}+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\left(-\frac{1}{4}+\frac{1}{4}\right)+\cdots+\left(\frac{1}{n-2}-\frac{1}{n-1}\right)+\left(-\frac{1}{n}+\frac{1}{n}\right)-\frac{1}{n+1}.$$

We see that only the first and the last terms survive, while the rest are canceled in pairs (which explains the name "telescoping" method). Thus, the whole sum is simply equal to:

$$1-\frac{1}{n+1}=\frac{n+1}{n+1}-\frac{1}{n+1}=\frac{(n+1)-1}{n+1}=\frac{n}{n+1}. \quad \square$$

If we want to add up *all* fractions $\frac{1}{n(n+1)}$, then no fractions will "survive" the cancellation except for the 1 in the front. A more formal approach would be to think of what number ("limit") the partial sums $\frac{n}{n+1}$ approach as n goes to infinity. Recalling and backtracking to a previous step above:

$$\frac{n}{n+1}=1-\frac{1}{n+1}.$$

As n increases, the denominator of $\frac{1}{n+1}$ increases to infinity, causing the fraction $\frac{1}{n+1}$ to go to 0. Therefore, the whole expression on the RHS goes to $1-0=1$. Thus, once again, the total sum of all fractions of the type $\frac{1}{n(n+1)}$ is 1:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\cdots+\frac{1}{n(n+1)}+\cdots=1. \quad \square$$

See if you can apply induction or the telescoping method to the following similar sums.

Problem 3. What are the following sums equal to:

(a) $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{n(n+2)}+\cdots=?$

(b) $\frac{1}{1\cdot 4}+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 6}+\frac{1}{4\cdot 7}+\frac{1}{5\cdot 8}+\cdots+\frac{1}{n(n+3)}+\cdots=?$

HINT: (a) $\frac{1}{1\cdot 3}=\frac{1/2}{1}-\frac{1/2}{3}$; (b) $\frac{1}{1\cdot 4}=\frac{1/3}{1}-\frac{1/3}{4}$.

Problem 4. Prove by induction the following formulas for any $n=1,2,3,\cdots$:

(a) $1+2+3+\cdots+n=\frac{n(n+1)}{2}$.

(b) $1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$.

(c) $1+3+5+7+\cdots+(2n-1)=n^2$.

3. Useful Formulas with Fractions and "Little" Algebra Truths

1. Definition. A *ring* R is a non-empty set with two binary operations called *addition* and *multiplication* and denoted by $(a,b) \overset{+}{\mapsto} a+b$ and $(a,b) \overset{\cdot}{\mapsto} a \cdot b = ab$ such that the following properties (*axioms*) are satisfied for any $a, b, c \in R$:

(1) $a+b$ is in R (closure under addition);
(2) $a+b=b+a$ (commutativity of addition);
(3) $(a+b)+c=a+(b+c)$ (associativity of addition);
(4) There is an element 0_R of R such that $a+0_R=0_R+a=a$ (zero element);
(5) For any a in R there is some a_1 of R such that $a+a_1=0_R$ (additive inverse);
(6) ab is in R (closure under multiplication);
(7) $(ab)c=a(bc)$ (associativity of multiplication);
(8) $a(b+c)=ab+ac$ and $(b+c)a=ba+ca$ (distributivity).

2. Examples of rings: The set of integers $\mathbb{Z}$, of rationals $\mathbb{Q}$ (these are all fractions a/b for integer a, b with $b \neq 0$), of reals $\mathbb{R}$, and of complex numbers $\mathbb{C}$; also the set of even integers E, of all integers divisible by 3, or of all integers divisible by any fixed integer k ($\neq 0$).

3. Examples of non-rings: The set of natural numbers $\mathbb{N}$, the set of odd integers O.

4. Definition. A *commutative* ring R is a ring with the following additional axiom:

(9) $ab=ba$ for all a, b in R (commutativity of multiplication).

5. Definition. A *field* F is a commutative ring with the additional axioms:

(10) For some element 1_R of R, $a \cdot 1_R = 1_R \cdot a = a$ for all a in R (multiplicative identity);
(11) For any a in R there is a_2 in R such that $aa_2 = a_2a = 1_R$ (multiplicative inverses);
(12) $1_R \neq 0_R$ (i.e., R consists of more than just one element).

6. Examples of fields: The set of rationals $\mathbb{Q}$, reals $\mathbb{R}$, and of complex numbers $\mathbb{C}$.

7. Examples of non-fields: The set of integers $\mathbb{Z}$, the set of even integers E, etc.

Note that the existence of additive inverses in rings allows us to define, in effect, a third operation, *subtraction*: $a-b=a+(-b)$ where $(-b)$ is the (unique) additive inverse of b. Using ring properties, one can multiply out to get:

8. Baby Binomial Theorem: $(a+1)^2 = a^2+2a+1$ for any a in a ring R.

In a field, the existence of multiplicative inverses allows us to define, in effect, a fourth operation, *division* by $a : b = a \cdot (b^{-1})$, where b^{-1} is the (unique) additive inverse of b. In the set of rationals $\mathbb{Q}$, $a \cdot (b^{-1})$ is denoted for short by $\frac{a}{b}$, which we know as the *fraction* "a divided by b". All laws or algebraic manipulations of fractions that we know are derived from the above axioms of $\mathbb{Q}$ as a field.

9. Universal Addition of Fractions: $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{ad+cb}{bd}$ for $b, d \neq 0$.

10. Splitting of Fractions:

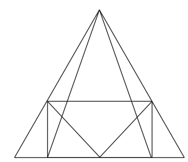
(a) $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$ and $\dfrac{a-b}{c} = \dfrac{a}{c} - \dfrac{b}{c}$ for $c \neq 0$.

(b) $\dfrac{1}{bd} = \dfrac{1}{d-b}\left(\dfrac{1}{b} - \dfrac{1}{d}\right)$ for $b, d \neq 0$ and $b \neq d$.

(c) $\dfrac{1}{bd} = \dfrac{1}{d+b}\left(\dfrac{1}{b} + \dfrac{1}{d}\right)$ for $b, d, \neq 0$ and $b \neq -d$, correspondingly.

4. From Combinatorial Geometry & Number Theory to Think About

Problem 5. How many triangles are there in the picture?



(5A) **(New version)** In a regular pentagon connect any two vertices. How about a regular hexagon?

(5B) **(Ultimate version 1)** Draw n lines in the plane so that no three intersect in the same point and no two are parallel. Into how many regions do these lines divide the plane?

(5C) **(Ultimate version 2)** In a regular n-gon connect any two vertices. How many triangles are formed this way?

(5D) **(Ultimate version 3)** Pick n points on a circle so that when you draw the segment between any two of these points, no three (or more) such segments intersect in the same place. Into how many pieces is the circle divided by all these segments?

Problem 6. Five teams participated in a basketball tournament. Each team played every other team exactly once. How many games were played in total?

(6A) **(New versions)** How many games were played if the teams were n instead of 5? How many games would be played in a tournament with n teams if each game involves exactly k teams?

(6B) **(Ultimate version)** How many games would be played in a tournament with n teams if each game can be played by any numbers of teams, from 1 to n, and all such games have been played?

Problem 7. If same digits correspond to same letters, replace the letters with digits so that the equation $\overline{xyzt} + \overline{xyz} + \overline{xy} + x = 2004$ is satisfied. (Here $\overline{xyzt}$ means that the four digits x, y, z, and t are put next to each other to form a 4-digit number, etc.)

(7A) **(Ultimate version)** If same digits correspond to same letters, replace the letters with digits so that the equation $xyzt + xyz + xy + x = 2004$ is satisfied. (Here $xyzt$ means that the product of the four digits x, y, z, and t, etc.)