Bulgarian Math Olympiads with a Challenge Twist

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Berkeley Math Circle – Beginners Group

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Tasks throughout this session. Harder versions[1] of problems from last time appear below. For each version:

(A) Solve the problem. If possible, solve it in more than one way.
(B) Write down the problem-solving techniques (PST's) and theorems you used to solve the problem. What made the solution(s) possible, i.e., what was the key idea(s)?

1. From Combinatorics[2]

Problem 1. A 4×4 square table is filled with the numbers $1, 2, 3$, and 4 in such a way that every number appears once in every row, column, and diagonal. What number is A?

1	2	3	4
A			
	1	4	3

(1A) **(New version)** Can you remove one of the initial seven numbers and still be able to *uniquely* fill in the whole table?

(1B) **(Ultimate version)** What is the *minimal* initial set of numbers (anywhere in the table) that will allow you to fill in the table in a *unique* way? Why?

Problem 2. Five teams participated in a basketball tournament. Each team played every other team exactly once. How many games were played in total?

(2A) **(New versions)** How many games were played if the teams were n instead of 5? How many games would be played in a tournament with n teams if each game involves exactly k teams?

(2B) **(Ultimate version)** How many games would be played in a tournament with n teams if each game can be played by any numbers of teams, from 1 to n, and all such games have been played?

[1]Some of the new and "ultimate" versions are really hard (if not *open*!) problems, so don't try to solve everything. In fact, concentrate only on problems where you can make some progress. Solving even one such extended version is an accomplishment. View this set of problems as an overarching goal for this semester or this year at the math circle, and come back to them whenever you have an opportunity. Ask questions, investigate, and work with others on their solutions. In other words, indulge a long-term *research* project.

[2]The math areas are assigned loosely to the problems. A specific problem may belong to a different area of math depending on what solution you find or how you interpret and apply the problem.

2. From Number Theory

Problem 3. The 5-digit number $\overline{21 * 36}$ is divisible by 3. What is the product of all possible values of $*$?

(3A) **(New version)** If you put the numbers 1, 2, 3, ..., 2014 next to each other to form a HUUUUGE number, and then you put one * between some two consecutive digits in this number, what is the product of all possible values of *?

(3B) **(Ultimate version)** What is the criterion for divisibility by 3? By 9? By 11? By 7? Why? Can you prove these criteria? Can you replace 3 by each of 9, 11 and 7 in the original or new versions and still be able to solve the problem?

Problem 4. Several books must be packed. If in each packet we put 3, or 4, or 5, or 6 books, there will always be 1 unpacked book; but if in each packet we put 7 books, then all books will be packed. What is the smallest possible number of books that we need to pack?

(4A) **(New version)** Several books must be packed. If in each packet we put 3, or 4, or 5, or 6, or 7, or 8, or 9, or 10 books, there will always be one unpacked book; but if in each packet we put 11 books, then all books will be packed. What is the smallest possible number of books that we need to pack?

(4B) **(Ultimate version)** How about if, for some *prime* number p, we extend to p books instead of 11? What if we replace $3, 4, 5, \ldots, p-1$, and p books by random positive integers a_1, a_2, ..., a_{k-1}, and a_k? For which such collections $\{a_1, a_2, \ldots, a_k\}$ is there a solution to the problem? Why? Can you describe *all* such solutions?

3. From Sums and Series (Finite and Infinite Summations)

Problem 5. What is the sum $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}$ equal to?

(5A) **(New versions)** What are the following sums equal to:

- $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\cdots+\frac{1}{2013\cdot 2014}=\,?$
- $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{2012\cdot 2014}=\,?$

(5B) **(Ultimate versions)** What are the following sums equal to:

- $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\cdots+\frac{1}{n(n+1)}+\cdots=\,?$
- $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{n(n+2)}+\cdots=\,?$
- $\frac{1}{1\cdot 4}+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 6}+\frac{1}{4\cdot 7}+\frac{1}{5\cdot 8}+\cdots+\frac{1}{n(n+3)}+\cdots=\,?$

Problem 6. A handyman can complete an order in 5 hours, and his apprentice in 2 more hours. The two worked together 1 hour. What part of the order remained unfinished?

(6A) **(New version)** A handyman can complete an order in 1 hour, and he has *infinitely* many apprentices named A_1, A_2, A_3, $\ldots$, A_n, and so on. The nth apprentice works half as fast as the previous apprentice (where the "0th apprentice" is the handyman himself). If all of the apprentices and the handyman work together for an hour, how many orders will they complete? Why?

(6B) **(Ultimate version)** In the new version with *infinitely* many apprentices, suppose the nth apprentice can complete an order in 1 more hour than the previous apprentice. For any M of orders, can you select *finitely* many apprentices who can complete the M orders in 1 hour only? Why?

Problem 7. If $A = 1 + 3 + 5 + \cdots + 2001 + 2003$ and $B = 2004 + 2002 + \cdots + 6 + 4 + 2$, how much is the difference $B - A$?

(7A) **(New versions)** How much is A? B? Why? What if the sum in A goes until any odd number $n = 2k - 1$ instead of 2003 and the sum in B starts from the next, even number $n + 1 = 2k$ instead of 2004?

(7B) **(Ultimate versions)** If $A = 1^2 + 3^2 + 5^2 + \cdots + 2003^2$ and $B = 2004^2 + 2002^2 + \cdots + 6^2 + 4^2 + 2^2$, how much is the difference $B - A$? How about if we replace 2003 and 2004 by any two consecutive odd and even integers $2k - 1$ and $2k$? Can you find individually A and B in each case?

4. From Areas in Geometry

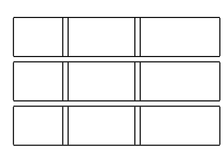
Problem 8. In $\triangle ABC$, M is the midpoint of BC, L is the midpoint of AM, and K is the midpoint of AB. If the area of $\triangle ABC$ is $36\,\text{cm}^2$, how much is the area of $\triangle KBL$?

(8A) **(New version)** Prove the formula for the area of a triangle that is used in the solution to the original problem.

(8B) **(Ultimate versions)** For any natural number n, starting from $\triangle ABC$ you can add consecutively *only midpoints* of segments that you can make by connecting already existing points on your picture (i.e., by connecting vertices or already existing midpoints). Construct a problem that results in a small triangle with area $1/2^n$ of the area of the original $\triangle ABC$. Prove that your construction works.

Now pick any triangle formed by three points in your picture: what fraction of the original area can the area of this triangle possibly be?

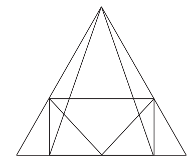
Problem 9. In a rectangular garden with length 40m and width 25m there are four alleys with width 5m each, as shown on the diagram. What is the area of the alleys?



(9A) **(New version)** In a rectangular garden with length A and width B there are m horizontal and n vertical alleys with width d each. What is the area of the alleys?

(9B) **(Ultimate version)** To the original 2 horizontal and 2 vertical alleys, add two alleys parallel to one of the diagonals and two alleys parallel to the other diagonal of the rectangle. What is the maximal and minimal total area of the alleys? Why?

5. From Combinatorial Geometry



Problem 10. How many triangles are there in the picture?

(10A) **(New version)** In a regular pentagon connect any two vertices. How about a regular hexagon?

(10B) **(Ultimate version 1)** In a regular n-gon connect any two vertices. How many triangles are formed this way?

(10C) **(Ultimate version 2)** Pick n points on a circle so that when you draw the segment between any two of these points, no three (or more) such segments intersect in the same place. Into how many pieces is the circle divided by all these segments?

6. From Number Theory/Combinatorics/Algebra/Logic

Problem 11. The product of 100 natural numbers equals 100. Find all possible sums of these 100 numbers.

(11A) **(New version)** The product of 100 natural numbers equals 100. How many such 100-tuples are there if the order of the numbers in a 100-tuple matter? How about if the order does not matter?

(11B) **(Ultimate version)** Given a natural number n, the product of n natural numbers equals n. How many such n-tuples are there if the order matters? How about if the order does not matter?

Problem 12. How many 4-digit numbers have their digits add up to 4?

(12A) **(New version)** How many 5-digit numbers have their digits add up to 5?

(12B) **(Ultimate version)** How many n-digit numbers have their digits add up to n?

Problem 13. If same digits correspond to same letters, replace the letters with digits so that the equation $\overline{xyzt} + \overline{xyz} + \overline{xy} + x = 2004$ is satisfied. (Here $\overline{xyzt}$ means that the four digits x, y, z, and t are put next to each other to form a 4-digit number, etc.)

(13A) **(New version)** Replace the letters x, y, z, and t with *different* digits so that the equation $xyzt + xyz + xy + x = 2004$ is satisfied. (Here $xyzt$ means that the digits x, y, z, and t are multiplied, etc.)

(13B) **(Ultimate version)** What numbers in place of 2004 would yield a *unique* answer in the original problem #13? How about in #13A?

Problem 14. Put the 7 pieces on the right next to each other in the 7 places to form an equation.

1	2	4	5	+	×	=

(14A) **(New version)** How many answers are there in the original problem?

(14B) **(Ultimate version)** Can you think of other sets of pieces that would yield an interesting problem?

7. Examples of Ideas and Problem Solving Techniques

During the session on September 3 we solved the following problems in several ways and discussed strategies and PST's in their solutions.

7.1. **Problem 2.** Five teams participated in a basketball tournament. Each team played every other team exactly once. How many games were played in total?

PST 1. Arrange the teams in one line and count the number of games, making sure that you are not to overcounting any games.

For 5 teams, this calculation boils down to $4+3+2+1=10$. $\square$

When the problem is generalized to n teams, the resulting sum is $1+2+3+\cdots+(n-1)$, which requires a different argument to be calculated in a concise form.

PST 2. Count the number of games played by each team *by double-counting* the games – once for one of the teams and another time for the other team, and then divide by 2.

For 5 teams, this yields again

$$\frac{\big(5 \text{ teams}\big)\cdot\big(4 \text{ games each}\big)}{2} = 10. \quad \square$$

Fortunately, this PST works for any n teams too:

$$\frac{\big(n \text{ teams}\big)\cdot\big((n-1) \text{ games each}\big)}{2} = \frac{n(n-1)}{2}\cdot \quad \square$$

As a result, we proved the following:

Theorem 1 (Gauss Sum)**.** The sum of the first n natural numbers is given by the formula

$$1+2+3+\cdots+n = \frac{n(n-1)}{2}\cdot$$

Both sides of this formula counted differently the same quantity: the number of games to be played among n teams in a tournament.

PST 3. To prove a formula LHS = RHS, find a quantity that equals both sides and calculate it in two different ways.

In another attempt to prove this formula, we drew an $n \times n$ table to record the scores of all games. Along the diagonal we wrote X's, since no team plays against itself. We were not interested in the outcomes of the games; so what was written otherwise in the table did not matter. What mattered were the number of non-diagonal cells: for every cell under the diagonal there was a cell above the diagonal that referred to the same game, e.g., a cell for the game between teams i and j and a cell for the game between teams j and i. Thus, the number of games would the half of the non-diagonal cells:

$$\frac{n^2 - n}{2} = \frac{n(n-1)}{2},$$

where n^2 are the cells of the whole table, the subtracted n are the diagonal cells, and the division by 2 offsets the double-counting of the games. □

PST 4. To calculate a positive integer quantity, find a table whose cells approximate this quantity. Start overcounting the quantity by the number of cells in the table, and then subtract and divide as necessary to eliminate unwanted cells or cells that have been counted multiple times.

In our final approach to solve the problem, we realized that the number of games corresponds to the number of pairs of teams that we can select from the given n teams. There is a quantity in combinatorics which counts even more general objects:

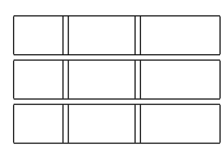
Theorem 2 (Binomial Theorem)**.** The number of ways to choose k objects out of n objects is the *binomial coefficient* $\binom{n}{k}$ (read "n choose k") and is calculated by the formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ for any } n, k \geq 0.$$

Applying this formula for $k = 2$ yields again

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{1 \cdot 2 \cdots (n-2) \cdot (n-1) \cdot n}{(1 \cdot 2)(1 \cdot 2 \cdots (n-2))} = \frac{(n-1)n}{2}.$$

7.2. **Problem 9.** In a rectangular garden with length 40m and width 25m there are four alleys with width 5m each, as shown on the diagram. What is the area of the alleys?



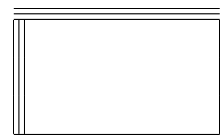
In our first attempt, we applied an over-counting PST: we added up the areas of all horizontal and all vertical alleys, and then subtracted the area of the overlap, which consisted of 4 little squares:

$$(2 \cdot 40 \cdot 5 + 2 \cdot 25 \cdot 5) - 4 \cdot 5^2 = 400 + 250 - 100 = 550\,\text{cm}^2. \quad \square$$

PST 5. To calculate the size of the union $A \cup B$ of two sets A and B, add up their sizes and subtract the size of their intersection $A \cap B$:

$$\#(A \cup B) = \#A + \#B - \#(A \cap B).$$

In our second attempt, we translated (geometrically shifted) all alleys within the garden so that the horizontal alleys were as far up as possible, and the vertical alleys were as far to the left as possible (without landing on the top of each other). This did not change the area of the alleys. Then we calculated the total area of the garden and subtracted the remaining, non-alley area:



$$40 \cdot 25 - (40 - 10) \cdot (25 - 10) = 1000 - 30 \cdot 15 = 1000 - 450 = 550\,\text{cm}^2. \quad \square$$

PST 6. To calculate the area of geometric figures, *translate* some of the figures to more convenient positions in the picture, without changing the area in question. Such convenient positions are often *extremal* positions, i.e., at some end of the full picture.

PST 7. To calculate a quantity, calculate everything and then subtract the *complement*:

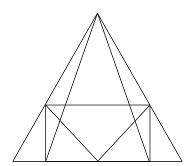
$$\text{a part} = \text{everything} - \text{the rest}.$$

In our final attempt, we directly calculated the area of the alleys after translation, using two rectangular parts:

$$40 \cdot 10 + (25 - 10) \cdot 10 = 400 + 150 = 550\,\text{cm}^2. \quad \square$$

PST 8. To calculate the area of a figure, *split it conveniently* into two or more simpler figures (e.g., rectangles), whose individual areas can be easily calculated.

7.3. **Problem 10.** How many triangles are there in the picture?



PST 9. To count how many figures of a certain type are there:

(9a) Count first those that are made of one piece, then those made of two pieces, then of three pieces, and so on.

(9b) Reconstruct the picture from scratch, adding incrementally simple parts (such as segments or small triangles) and counting how many more figures of the wanted type are added at each step.

Using PST 9b, we started from the biggest triangle and kept adding segments, counting along the way and trying to use the symmetry of the picture as much as possible:

$$1 + 2 + 3 + 1 + 1 + 4 + 2 + 2 + 3 + 3 + 4 + 5 = 31. \quad \square$$

Check that with PST 9a you will also get 31 triangles. Alternatively, start from the little triangle in the left bottom corner, and incrementally build the picture by adding one little *triangle* at a time.

7.4. **Problem 13.** If same digits correspond to same letters, replace the letters with digits so that the equation $\overline{xyzt} + \overline{xyz} + \overline{xy} + x = 2004$ is satisfied. (Here $\overline{xyzt}$ means that the four digits x, y, z, and t are put next to each other to form a 4-digit number, etc.)

We started solving this problem by

PST 10. To preserve the value of a sum of integers (or decimals), you can move digits around keeping track of the place values of these digits.

In our problem, we have an x in the ones' place, in the tens' place, in the hundreds' place, and in the thousands' place, so we can move the x's around to form $\overline{xxxx}$, thereby preserving the total contribution of x to our expression:

$$\overline{xyzt} + \overline{xyz} + \overline{xy} + x = \overline{xxxx} + \overline{yyy} + \overline{zz} + t = 2004.$$

Then we factored out x, y, and z:

$$x\overline{1111} + y\overline{111} + z\overline{11} + t\overline{1} = 2004.$$

PST 11. If two integer expressions are equal, then they have *the same remainder* r when divided by any integer d. So, choose a convenient d to divide by and compare remainders on both sides. A *convenient divisor* d is usually one that divides one or more of the involved numbers.

For example, $d = 11$ divides 1111 and 11, while its remainder from 111 is 1, and from 2004 is 2 (check this!). Thus, we divided both sides of our equation by 11 and equated the remainders: $y + t$ must have a remainder of 2 when divided by 11 (why?). Since y and t are digits, their sum can only be 2 or 13 (why?), i.e., $y + t = 2$ or $y + t = 13$. This restricts our choices, but does not yet give us the values for y and t.

Another idea to try is a *extremal approach*:

PST 12. If the values of some variables in an expression are restricted (e.g., they are digits), use the minimum and/or maximum values and *construct inequalities* to calculate how small or large an expression can be. This may *eliminate values* of the variables.

In our problem, all variables x, y, z, t are between 0 and 9. To add up to 2004, x cannot be 2 or more (why?), hence $x = 0, 1$. The remaining variables can contribute at most

$$y\overline{111} + z\overline{11} + t\overline{1} \leq 999 + 99 + 9 = 1107.$$

Hence x must contribute something too, i.e., $x \neq 0$. We are left with $x = 1$.

PST 13. Once you *acquire information* about some of the unknowns, *plug it in* to restrict the number of unknowns and simplify the problem.

Thus, plugging in $x = 1$ leaves us with $y\overline{111} + z\overline{11} + t\overline{1} = 2004 - 1111 = 893$. Using the extremal approach,

$$z\overline{11} + t\overline{1} \leq 99 + 9 = 108$$

so that $y\overline{111} \geq 893 - 108 = 785$, from which $y \geq 8$ (why can't $y = 7$?), i.e., $y = 8, 9$. But $y \neq 9$ because $999 > 893$, so we are left with $y = 8$. Recalling that $y + t = 2$ or 13, we arrive at $8 + t = 13$, i.e., $t = 5$. Finally, $z\overline{11} = 893 - 888 - 5 = 0$, i.e., $z = 0$. Checking,

$$1111 + 888 + 00 + 5 = 2004,$$

and we have a *unique* solution $\overline{xyzt} = 1805$. □