MATHEMATICAL INDUCTION

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1. Cookie Monster starts a cookie marathon. He eats one cookie on Monday, two cookies on Tuesday, four on Wednesday. Every day he eats twice as many cookies as he ate the previous day. How many cookies did he eat on the first Sunday? How many cookies did he eat for the first week? in two weeks? Can you calculate the total number of cookies eaten by Cookie Monster for n days of this cookie marathon?

2. You have 9 coins, and you know that one coin is fake. It weights less than a real coin. You have a balance. What is the minimal number of weighings needed for finding the fake coin? What if you have 81 coins and again one is fake? Can you solve the problem for any number of coins if you know that there is exactly one fake coin and it is lighter than real coins?

3. The tower of Hanoi is a puzzle containing three rods and several disks with the hole in the middle so that a disk can slide down a rod. All disks have different sizes. In the beginning all disks are on one rod in the descending order, with the larges disk at the bottom and the smallest one at the top. Your task is to move all the disks to another rod obeying the following rules:

By one move one may take one disk from the top of one rod and put it onto another rod.

It is forbidden to put a larger disk on top of a smaller disk.

(a) Show that you can perform the task for any number of disks.

(b) What is the minimal number of moves you need to perform this task?

4. Several pirates want to divide the loot between themselves. Every pirate believes that he can divide the loot in any number of equal parts but he does not trust that his fellow pirates can do the same. A pirate is happy if he believes that he got not less than his fare share. Say, if there are four pirates, then a pirate is happy if he believes that his share is at least quarter of the loot. Is it possible for them to agree how to divide the loot?

For instance, if there are only two pirates, then one of the pirates can divide the loot into two equal parts (in his opinion), and the other can choose the part that he believes is larger.

Can you suggest a scheme of dividing the loot among three pirates so that every pirate is happy with his share? Can you do the same for any number of pirates?

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5. A chess board has one corner square cut out. Can you cover this board by L shape pieces made out of three squares without overlapping? Can you do the same for any $2^n \times 2^n$ board with one corner cut out?

6. Seven wise men are sitting in a train compartment. Every men has a dirty face but can not see himself. A ticket collector comes into the compartment and says "Oh, some of you need to wash their faces". There is no water on the train and to wash his face a wise man should get out at a station. Wise men did not move for some time, but suddenly at some station they all run out of the train. How many stations did they skip after the visit of the ticket collector before going out? How does the answer depend on the number of wise men?

7. One draws n lines on the plane so that any two lines meet at one point and no three lines meet at the same point. Into how many parts these lines divide the plane? Start with $n = 2, 3, 4$. Make a conjecture. Then prove it.

8. In the situation of the previous problem show that you can color the parts of the plane in red and blue so that two adjacent parts have different color.

9. Show that the sum of the first n odd numbers equals n^2.

10. Show that the sum of the first n perfect cubes is a perfect square. For instance, $1^3 + 2^3 + 3^3 = 36 = 6^2$.

11. A long rectangular piece of greed paper of size $2 \times n$ is covered by rectangular 1×2 domino pieces. In how many ways one can do it?

12. Consider the Fibonacci sequence $1, 2, 3, 5, 8, \ldots$. In this sequence a number is the sum of two previous numbers. Prove that the difference of the square of a Fibonacci number and the product of the previous and the next Fibonacci number is plus or minus 1.

13. A positive integer n is called prime if it is bigger than 1 and all its divisors are 1 and n. Prove that any number $n > 1$ can be written as a product of prime numbers.

14. At a table tennis tournament every player plays once with any other player. There were no ties. A player A gets a prize if for any other player B either A beats B or A beats some other player who beats B. Prove that at least one player gets a prize. Is it possible that more than one player gets a prize?