Graph theory I

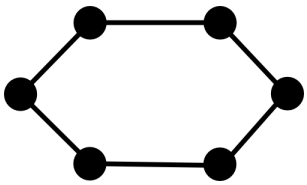
Berkeley Math Circle, Beginners

October 1, 2013

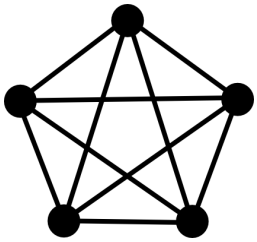
Introduction

A *graph* consists of *vertices* and *edges*. You can think of vertices as points, and edges are lines that connect some pairs of points. Here are some examples:

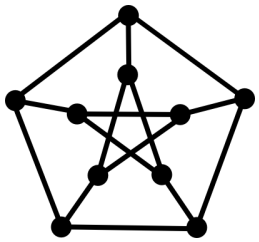
(a) A path on 4 vertices.



(b) A cycle on 6 vertices.



(c) The complete graph on 5 vertices.



(d) The Petersen graph.

Figure 1: Examples of graphs.

Although the figures above are all nice and symmetric, we do not care how a graph is drawn—the only thing that matters is whether there exists an edge between a pair of vertices or not. An edge can be a straight line, a curve, or anything you want, as long as it connects the two vertices. A pair of vertices which are connected by an edge are called *adjacent*.

We will only be considering *simple* graphs: ones with no self-loops, and no multiple edges between two vertices. Also, we will only be considering *undirected* graphs: there is no distinction between the two vertices connected by an edge. (There are also *directed* graphs, where the edges point from one vertex to another.)

The *degree* of a vertex is the number of edges going out of it.

Introductory problems

1. **(Number of edges)**
   (a) Take a path with 5 vertices. How many edges does it have? How about a path with 6 vertices? 2013 vertices? n vertices?
   (b) Take a cycle with 5 vertices. How many edges does it have? How about a cycle with 6 vertices? 2013 vertices? n vertices?
   (c) Take a complete graph with 5 vertices. How many edges does it have? How about a complete graph with 6 vertices? 2013 vertices? n vertices?
2. **(Number of graphs)**
   (a) How many graphs are there with exactly 3 vertices? Draw all of them!

(b) How about with exactly 4 vertices? Draw all of them!

3. **(Degrees)**

   (a) Take a path with 5 vertices. What are the degrees of the vertices? How about a path with 6 vertices? 2013 vertices? n vertices?

   (b) Take a cycle with 5 vertices. What are the degrees of the vertices? How about a cycle with 6 vertices? 2013 vertices? n vertices?

   (c) Take a complete graph with 5 vertices. What are the degrees of the vertices? How about a complete graph with 6 vertices? 2013 vertices? n vertices?

4. **(Sum of degrees)**

   (a) Take a path with 5 vertices. What is the sum of the degrees of the vertices? How about a path with 6 vertices? 2013 vertices? n vertices?

   (b) Take a cycle with 5 vertices. What is the sum of the degrees of the vertices? How about a cycle with 6 vertices? 2013 vertices? n vertices?

   (c) Take a complete graph with 5 vertices. What is the sum of the degrees of the vertices? How about a complete graph with 6 vertices? 2013 vertices? n vertices?

   (d) **(Handshaking lemma)** Do you notice anything about the sum of the degrees? What is its parity? Can you prove that your observation is true for every graph?

Colorability

A *vertex coloring* of a graph is an assignment of colors to each vertex of the graph in such a way that no two adjacent vertices have the same color. An interesting question in general is to find the minimum number of colors needed to color a graph. This minimum number is called the *chromatic number* of the graph.

6. **(Coloring simple graphs)**

   (a) Take a path with 5 vertices. How many colors do you need to color it? How about a path with 6 vertices? 2013 vertices? n vertices?

   (b) Take a cycle with 5 vertices. How many colors do you need to color it? How about a cycle with 6 vertices? 2013 vertices? n vertices?

   (c) Take a complete graph with 5 vertices. How many colors do you need to color it? How about a complete graph with 6 vertices? 2013 vertices? n vertices?

7. **(Petersen graph)** How many colors do you need to color the Petersen graph (see page 1)?

8. **(An upper bound)** Suppose I tell you that I will give you a graph on 5 vertices (but I do not tell you which one), and that you will need to color it. How many colors do you need to bring in the worst case? What if my graph has 10 vertices? n vertices?

9. **(A lower bound)** Suppose I tell you that I will give you a graph that you need to color, and this graph has the property that there exist 3 vertices all of whom are connected with each other. How many colors do you need at least? What if there are 6 vertices all of whom are connected with each other? n vertices?

Ramsey-type problems

11. Consider a party of 5 people. Is it true that there is always a group of three who either all know each other or are all strangers to each other? What if the party consists of 6 people?