Probability and Random Walks 1
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A **random variable** is a variable X that takes certain values with different probabilities.

A **probability distribution** for a random variable tells you the probabilities that a random variable has a certain value.

Problem 1: Find probability distributions for the following random variables:

(a) The result of a single coin flip, where heads is 1 and tails is 0.

(b) The sum of two coin flips.

(c) The sum of three coin flips.

(d) The sum of two die rolls. What is the most likely sum?

(e) The sum of three rolls of a four-sided die.

(f) The sum of three flips of a biased coin: It has 2/3 chance of coming up heads and 1/3 chance of coming up tails.

Problem 2: What's the most likely outcome for two rolls of an $n-$sided die with sides numbered $1, 2, \ldots, n$? What about 3? k?

In the **random walk** problem, an overly excited puppy starts at position 0. Then, with probability 1/2 he jumps to the right (position 1) and with probability 1/2 he jumps to the left (position -1). He continues to jump right or left with probability 1/2.

Problem 3: Find the probability distribution for the puppy's position after 0 jumps, 1 jump, 2 jumps, 3 jumps, and 4 jumps. You can solve this in the following way: For a given number of jumps, list all of the puppy's possible moves. For example, for 2 jumps, the moves are RR, RL, LR, and LL. Each of these has probability 1/4. Then, find the ending position resulting from each of these series of jumps.

Do you notice a pattern in the probability distributions? Make a conjecture about the probability distribution after n jumps. Can you prove your conjecture?

Problem 4: In the **bounded random walk**, if the puppy ever reaches position -1 or position 2, he must stop.

(a) Find the probability distribution after 0 jumps, 1 jump, 2 jumps, 3 jumps and 4 jumps.

(b) Can you find a pattern? What happens after a large number of jumps? Can you prove it?

(c) Try this again, but instead the puppy is forced to stop on positions -1 and 3. What about -2 and 2?

If you don't want to enumerate the puppy's paths for the bounded walk, here's another way to do it: At each stage, the puppy (not at the ends) has fifty percent chance to jump right and fifty percent chance to jump left. So to get from one stage to the next: Replace each position with 1/2 of the number to the left and 1/2 of the number to the right (if those positions are not the ends). This is the basic idea behind **Markov chains**.

Problem 5: Investigate a random walk on a circle. Start with a circle consisting of 4 points. What is the probability distribution after 0 jumps, 1 jump, 2 jumps, 3 jumps and 4 jumps? What about after a large number of jumps? Can you prove it?

Problem 6: (Gambler's Ruin) Show that, no matter where the puppy starts, he will eventually hit position 0 with probability 1. More precisely: If the puppy starts at position $N > 0$ and undergoes a random walk where he has to stop if he reaches 0, then the probability distribution approaches 1 at position 0. Hint: Use the pattern you discovered in problem 4.

This is called "Gambler's Ruin" because it implies that, if a gambler plays a fair game against a bank with infinite wealth, the gambler will always eventually go broke.

Problem 7: For an unbounded random walk starting at position 0, find the probability that the puppy returns to position 0 for the first time after exactly 2 steps? 4 steps? $2n$ steps?

Problem 8: Now think about a random walk in two dimensions: At each step, the puppy jumps north, south, east or west each with probability 1/4. Find the probability distributions after 0 jumps, 1 jump, etc. If you draw a square boundary, where the puppy is forced to stop when he hits any of those points, what are the eventual probabilities that the puppy stops at a given boundary point?