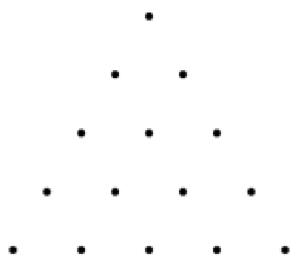
Bay Area Mathematical Olympiad

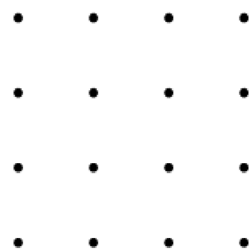
BAMO-8 Exam

http://www.bamo.org/archives

BAMO Preparation
at the Berkeley Math Circle-Beginners
with Zvezdelina Stankova, BMC Director
February 11, 2014

A How many different sets of three points in this equilateral triangular grid are the vertices of an equilateral triangle? Justify your answer.





1 A square grid of 16 dots (see the figure) contains the corners of nine 1×1 squares, four 2×2 squares, and one 3×3 square, for a total of 14 squares whose sides are parallel to the sides of the grid. What is the smallest possible number of dots you can remove so that, after removing those dots, each of the 14 squares is missing at least one corner?

Justify your answer by showing both that the number of dots you claim is sufficient and by explaining why no smaller number of dots will work.

1 Call a year *ultra-even* if all of its digits are even. Thus 2000, 2002, 2004, 2008, and 2008 are all ultra-even years. They are all 2 years apart, which is the shortest possible gap. 2009 is not an ultra-even year because of the 9, and 2010 is not an ultra-even year because of the 1.

(a) In the years between the years 1 and 10000, what is the longest possible gap between two ultra-even years? Give an example of two ultra-even years that far apart with no ultra-even years between them. Justify your answer.

(b) What is the second-shortest possible gap (that is, the shortest gap longer than 2 years) between two ultra-even years? Again, give an example, and justify your answer.

A A set of identical square tiles with side length 1 is placed on a (very large) floor. Every tile after the first shares an entire edge with at least one tile that has already been placed.

- What is the largest possible perimeter for a figure made of 10 tiles?
- What is the smallest possible perimeter for a figure made of 10 tiles?
- What is the largest possible perimeter for a figure made of 2011 tiles?
- What is the smallest possible perimeter for a figure made of 2011 tiles?

Prove that your answers are correct.

A Hugo plays a game: he places a chess piece on the top left square of a 20×20 chessboard and makes 10 moves with it. On each of these 10 moves, he moves the piece either one square horizontally (left or right) or one square vertically (up or down). After the last move, he draws an X on the square that the piece occupies. When Hugo plays the game over and over again, what is the largest possible number of squares that could eventually be marked with an X? Prove that your answer is correct.

A We write $\{a,b,c\}$ for the set of three different positive integers a, b, and c. By choosing some or all of the numbers a, b and c, we can form seven nonempty subsets of $\{a,b,c\}$. We can then calculate the sum of the elements of each subset. For example, for the set $\{4,7,42\}$ we will find sums of 4, 7, 42, 11, 46, 49, and 53 for its seven subsets. Since 7, 11, and 53 are prime, the set $\{4,7,42\}$ has exactly three subsets whose sums are prime. (Recall that prime numbers are numbers with exactly two different factors, 1 and themselves. In particular, the number 1 is *not* prime.)

What is the largest possible number of subsets with prime sums that a set of three different positive integers can have? Give an example of a set $\{a,b,c\}$ that has that number of subsets with prime sums, and explain why no other three-element set could have more.

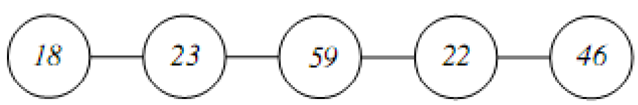
2 The *Fibonacci sequence* is the list of numbers that begins 1, 2, 3, 5, 8, 13 and continues with each subsequent number being the sum of the previous two.

Prove that for every positive integer n, when the first n elements of the Fibonacci sequence are alternately added and subtracted, the result is an element of the sequence or the negative of an element of the sequence. For example, when $n = 4$ we have

$$1 - 2 + 3 - 5 = -3,$$

and 3 is an element of the Fibonacci sequence.

B Five circles in a row are each labeled with a positive integer. As shown in the diagram, each circle is connected to its adjacent neighbor(s). The integers must be chosen such that the sum of the digits of the neighbor(s) of a given circle is equal to the number labeling that point. In the example, the second number $23 = (1+8)+(5+9)$, but the other four numbers do not have the needed value.



What is the smallest possible sum of the five numbers? How many possible arrangements of the five numbers have this sum? Justify your answers.

B Answer the following two questions and justify your answers:

(1) What is the last digit of the sum $1^{2012} + 2^{2012} + 3^{2012} + 4^{2012} + 5^{2012}$?

(2) What is the last digit of the sum $1^{2012} + 2^{2012} + 3^{2012} + 4^{2012} + \cdots + 2011^{2012} + 2012^{2012}$?

2 Consider a 7×7 chessboard that starts out with all the squares white. We start painting squares black, one at a time, according to the rule that after painting the first square, each newly painted square must be adjacent along a side to only the square just previously painted. The final figure painted will be a connected "snake" of squares.

(a) Show that it is possible to paint 31 squares.

(b) Show that it is possible to paint 32 squares.

(c) Show that it is possible to paint 33 squares.

B A *clue* "k digits, sum is n" gives a number k and the sum of k distinct, nonzero digits. An *answer* for that clue consists of k digits with sum n. For example, the clue "Three digits, sum is 23" has only one answer: 6,8,9. The clue "Three digits, sum is 8" has two answers: 1,3,4 and 1,2,5.

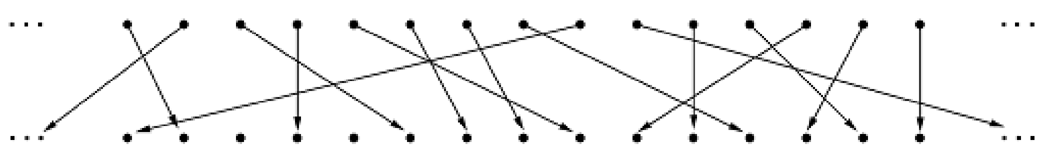
If the clue "Four digits, sum is n" has the largest number of answers for any four-digit clue, then what is the value of n? How many answers does this clue have? Explain why no other four-digit clue can have more answers.

B Let triangle ABC have a right angle at C, and let M be the midpoint of the hypotenuse AB. Choose a point D on line BC so that angle CDM measures 30 degrees. Prove that the segments AC and MD have equal lengths.

C Two infinite rows of evenly-spaced dots are aligned as in the figure below. Arrows point from every dot in the top row to some dot in the lower row in such a way that:

- No two arrows point at the same dot.
- No arrow can extend right or left by more than 1006 positions.

Show that at most 2012 dots in the lower row could have no arrow pointing to them.



D Laura won the local math olympiad and was awarded a "magical" ruler. With it, she can draw (as usual) lines in the plane, and she can also measure segments and replicate them anywhere in the plane. She can also divide a segment into as many equal parts as she wishes; for instance, she can divide any segment into 17 equal parts. Laura drew a parallelogram $ABCD$ and decided to try out her magical ruler. With it, she found the midpoint M of side CD, and she extended side CB beyond B to point N so that segments CB and BN were equal in length. Unfortunately, her mischievous little brother came along and erased everything on Laura's picture except for points A, M and N. Using Laura's magical ruler, help her reconstruct the original parallelogram $ABCD$: write down the steps that she needs to follow and prove why this will lead to reconstructing the original parallelogram $ABCD$.