Basics in Geometry I

at the Berkeley Math Circle-Beginners
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I. BAMO Super-Challenges

Problem 1 Let triangle ABC have a right angle at C, and let M be the midpoint of the hypotenuse AB. Choose a point D on line BC so that angle CDM measures 30 degrees. Prove that the segments AC and MD have equal lengths.

Problem 2 Laura won the local math olympiad and was awarded a "magical" ruler. With it, she can draw (as usual) lines in the plane, and she can also measure segments and replicate them anywhere in the plane. She can also divide a segment into as many equal parts as she wishes; for instance, she can divide any segment into 17 equal parts. Laura drew a parallelogram $ABCD$ and decided to try out her magical ruler. With it, she found the midpoint M of side CD, and she extended side CB beyond B to point N so that segments CB and BN were equal in length. Unfortunately, her mischievous little brother came along and erased everything on Laura's picture except for points A, M and N. Using Laura's magical ruler, help her reconstruct the original parallelogram $ABCD$: write down the steps that she needs to follow and prove why this will lead to reconstructing the original parallelogram $ABCD$.

Problem 3 Let C be a circle in the xy-plane with center on the y-axis and passing through $A = (0, a)$ and $B = (0, b)$ with $0 < a < b$. Let P be any other point on the circle, let Q be the intersection of the line through P and A with the x-axis, and let $O = (0, 0)$. Prove that $\angle BQP = \angle BOP$.

Problem 4 Let ABC be a triangle with D the midpoint of side AB, E the midpoint of side BC, and F the midpoint of side AC. Let k_1 be the circle passing through points A, D, and F; let k_2 be the circle passing through points B, E, and D; and let k_3 be the circle passing through C, F, and E. Prove that circles k_1, k_2, and k_3 intersect in a point.

Problem 5 Let ABC be a scalene triangle with the longest side AC. (A *scalene* triangle has sides of different lengths.) Let P and Q be the points on the side AC such that $AP = AB$ and $CQ = CB$. Thus we have a new triangle BPQ inside triangle ABC. Let k_1 be the circle *circumscribed* around the triangle BPQ (that is, the circle passing through the vertices B, P, and Q of the triangle BPQ); and let k_2 be the circle *inscribed* in triangle ABC (that is, the circle inside triangle ABC that is tangent to the three sides AB, BC, and CA). Prove that the two circles k_1 and k_2 are *concentric*, that is, they have the same center.

II. STARTING FROM THE BEGINNING: BASIC TRIANGLES AND QUADRILATERALS

Problem -7. Describe in words which triangles are called:

a) *equilateral*; b) *isosceles*; c) *scalene*; d) *right*; e) *acute*; f) *obtuse*; g) *right isosceles*.

Draw a couple of triangles of each type. You may use your ruler, right triangles, and protractor.

Problem -6. Let's think about the various types of triangles.

a) Is an *equilateral* triangle also *isosceles*? If yes, in how many ways? If no, why not?
b) Is an *isosceles* triangle *equilateral*? Can it happen sometimes? Explain.
c) Can a triangle be both *scalene* and *equilateral*? Both *scalene* and *isosceles*?
d) At most how many right angles can a triangle have? What are the other angles?
e) At most how many *obtuse* angles can a triangle have? What are the other angles?
f) How many *acute* angles can a triangle have at most? At least?*
g) Is there an *obtuse* triangle which is *isosceles*? How about an *equilateral obtuse* triangle?

Problem -5. Look at any isosceles triangle that you have drawn above. Measure its angles with a protractor. What do you notice? Repeat for any equilateral triangle drawn above. Explain.

Problem -4. Pick any triangle that you have drawn above, measure its three angles with a protractor and then add them up. What did you (approximately) get? What do you conclude the sum of the angles in a triangle is most likely to always be? Can you explain why this is so?

Problem -3. Draw the following Venn diagrams as elegantly as possible:

a) Venn diagram of all triangles according to their sides: equilateral, isosceles, scalene.
b) Venn diagram of all triangles according to their angles: acute, right, obtuse.
c) Venn diagram of all triangles according to their sides and angles: equilateral, isosceles, scalene, acute, right, obtuse.

Problem -2. Describe in words which figure is called a:

a) *square*; b) *rectangle*; c) *parallelogram*; d) *rhombus*; e) *trapezoid*; f) *deltoid*; g) *quadrilateral*.

(Warning: Some adults may disagree on what a trapezoid is! To set the record straight, and in view of what awaits for you in college, we shall say that "A *trapezoid* is a quadrilateral with at least one pair of parallel sides." So, is a parallelogram a trapezoid? Further, a *deltoid* is a kite, i.e., two adjacent sides are equal and the other two adjacent sides are also equal.)

Problem -1. Let's think about the various types of quadrilaterals. For each question, answer whether this happens: Always? Never? Sometimes? Why or why not?

a) Is a rectangle a square? Is a square a rectangle?
b) Is a parallelogram a rectangle? Is a rectangle a parallelogram?
c) Is a parallelogram a rhombus? Is a rhombus a parallelogram?
d) Is a trapezoid a parallelogram? Is a parallelogram a trapezoid?
e) Is a trapezoid a rectangle? Is a rectangle a trapezoid?
f) Is a square a trapezoid? Is a square a rhombus? Is a square a deltoid?
g) Is a rhombus a deltoid? Is a trapezoid a deltoid?

Problem 0. Using your answers from Problem -1, draw a *Venn diagram* that represents the relationships between all types of quadrilaterals discussed in Problem -2.