What Does the Future Hold for Restricted Patterns?[1]

by Zvezdelina Stankova

Berkeley Math Circle – Advanced Group

November 26, 2013

1. Basics on Restricted Patterns

1.1. **The primary object of study.** We agree to write a permutation τ of length k in one-row notation as $(a_1, a_2, \ldots, a_k)$ where $\tau(i) = a_i$ for $1 \leq i \leq k$. For $k < 10$ the commas can be supressed without causing confusion. As usual, S_n denotes the symmetric group on $[n] = \{1, 2, ..., n\}$.

The definition below is the fundamental one that underpins the whole area of restricted patterns:

Definition 1. Let τ and π be two permutations of lengths k and n, respectively. We say that π is τ-*avoiding* if there is no subsequence $i_{\tau(1)}, i_{\tau(2)}, ..., i_{\tau(k)}$ of $[n]$ such that $\pi(i_1) < \pi(i_2) < \ldots < \pi(i_k)$. If there is such a subsequence, we say that it is *of type* τ, and denote this by $\big(\pi(i_{\tau(1)}),\, \pi(i_{\tau(2)}),...,\, \pi(i_{\tau(k)})\big) \approx \tau$.

As one can see, without several examples worked out with a pencil and paper, this formal definition is not very insightful. To understand where restricted patterns really originated from, we draw on visual imagery and replace "one-dimensional" permutations by "two-dimensional" objects, matrices. In doing so, we shall violate the customary labeling of the top row as the "first" row of a matrix. Instead, we shall coordinatize our matrices from *the bottom left corner*, just like in a Cartesian coordinate system; thus, the origin will always be placed at the *bottom left corner* of the matrix (cf. Fig. 1) so that our first row will be the *bottom row*, and our first column will be (as usual) the leftmost column of a matrix. This is done in order to keep the resemblance with the "shape" of permutations, in other words, with their *graphs* as functions $\pi : [n] \to [n]$.

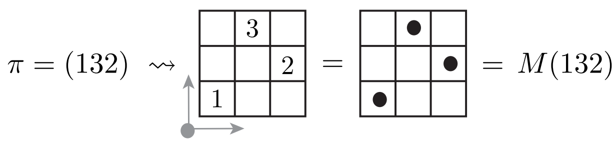

Figure 1. Permutation matrix $M(132)$

Here is a simple example to start with (cf. Fig. 1). The matrix $M(132)$ associated to $\pi = (132)$ is a 3×3 matrix with dots in cells $(1, 1)$, $(2, 3)$ and $(3, 2)$. Figure 2 displays the larger matrices $M(52687431)$, $M(3142)$ and $M(2413)$. In general,

[1]This is a paper in progress and contains some unfinished notes.

Definition 2. Let $\pi \in S_n$. The *permutation matrix* $M(\pi)$ is the $n \times n$ matrix M_n having a 1 (or a dot) in position $(i, \pi(i))$ for $1 \leq i \leq n$.

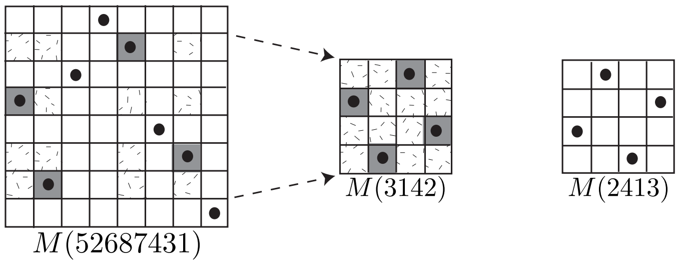

FIGURE 2. $(52687431) \notin S_8(3142)$, but $(52687431) \notin S_8(2413)$

As the reader has probably observed by now, a permutation matrix is nothing but an arrangement of n non-attacking rooks on an $n \times n$ board, called a *transversal* of the board with elements the "1's", or the "dots".

The original pattern-avoidance Definition 1 is designed in such a way that a permutation $\pi \in S_n$ contains a subsequence $\tau \in S_k$ exactly when the matrix $M(\pi)$ contains $M(\tau)$ as a submatrix. For instance, Figures 2a-b demonstrate that $\pi = (52687431)$ has a subsequence (6273) of type (3142) exactly because $M(52687431)$ has a 4×4 submatrix formed by the rows and columns of (6273) and identical to $M(3142)$. On the other hand, it is not hard to convince yourself that no submatrix identical to $M(2413)$ (cf. Fig. 2c) is contained in $M(52687431)$, and thus (52687431) avoids the permutation (2431).

We conclude that avoidance of *permutations* is an equivalent notion to avoidance among *permutations matrices*: permutation π avoids τ if and only if matrix $M(\pi)$ avoids $M(\tau)$, i.e. $M(\pi)$ does not contain a submatrix identical to $M(\tau)$.

1.2. **Interpretation of 231-avoidance.** We shall fix now one permutation τ and investigate the set of permutations of length n avoiding τ. This set is denoted by $S_n(\tau)$. To flesh out our understanding of pattern avoidance, let us describe one initial but nevertheless striking example: that of $S_n(231)$.

In [14] Knuth shows that $S_n(231)$ is precisely the set of *stack-sortable* permutations (cf. also [20]). To visualize the situation, imagine a train station with one main track, and one dead-end side track used for storing temporarily wagons. A cargo train is coming into the station from the right along the main track (cf. Fig. 3). Its four wagons (starting with the leading one) are numbered by "4", "1", "3" and "2". The goal is to rearrange the wagons so that the train leaves the station with wagons numbered in increasing order, "1", "2", "3", "4". We can use the side track to store as many wagons as we wish, however, at any time we can pull out only the most recently stored wagon onto the main track and we must push it immediately forward to join the sorted out train.

The reader can easily spot the solution to sorting out the given train "4132", and just as easily realize that this is impossible if the incoming train were labelled "3142" because of its

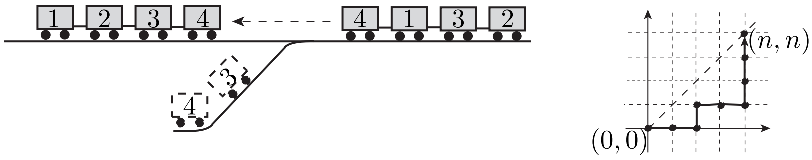

FIGURE 3. Stack-sortability, 231-avoidance, and Lattice paths

subsequence (342) of type (231). It turns out that containing the permutation (231) is the only obstruction to sorting out our trains. Thus, the permutations that can be sorted out in the above way, the so-called *stack-sortable* permutations, are precisely those that avoid (231).

Evidently, two different such permutations have different sorting algorithms. Furthermore, the sorting algorithm for each such permutation of length n is unique, encoded by binary strings of length $2n$, where "0" stands for "move into the stack", while "1" – for "move out of the stack". Since we can't move out of the stack more wagons than what is currently stored there, these binary strings can be thought of as properly parenthesizes expressions with "(" and ")" replacing corespondingly "0" and "1". Such expressions, on the other hand, are nothing else but lattice walks from the origin $(0, 0)$ to the point (n, n) that do not cross the diagonal $y = x$ and that are composed only of unit-length steps to the right or up. For instance, our train (4132) can be encoded as 00100111="(()(()))", which in turn is the lattice path shown in Figure 3b.

But as it is well-known, the *Catalan numbers* c_n also count exactly the same lattice paths![2] Making a full circle around, we conclude that

$$|S_n(231)| = c_n \text{ for all } n \geq 1. \tag{1}$$

For instance, there are $c_4 = 14$ trains of length 4 which can be sorted out. Equation (1) is the first elementary, yet non-trivial enumerative result on restricted patterns, which should have given the reader a flavor of the rich combinatorial possibilities in this field. Even though enumerating the various sets $|S_n(\tau)|$ is a worthy and challenging problem in itself (and we shall come to it in a later part of the paper), we mainly view it as a vehicle to solving a much more enticing puzzle.

1.3. **Wilf-equivalence.** Comparing two different permutations τ and σ in our setting naturally leads to comparing their associated S_n-subsets, $S_n(\tau)$ and $S_n(\sigma)$.

Definition 3. Two permutations τ and σ are *Wilf-equivalent*, denoted by $\tau \sim \sigma$, if they are equally restrictive on any length permutations, i.e.

$$|S_n(\tau)| = |S_n(\sigma)| \text{ for all } n \in \mathbb{N}.$$

[2]The Catalan numbers are given by: $c_n = \frac{1}{n+1}\binom{2n}{n} = \sum_{i=1}^{n} c_{i-1}c_{n-i}$, $c_0 = c_1 = 1$.

The classification of permutations up to Wilf-equivalence is the first classic and still far from resolved question in the field of restricted patterns. To get a feeling for it, let us investigate the first non-trivial situation, in S_3 (for there is nothing interesting to say in S_1 or S_2).

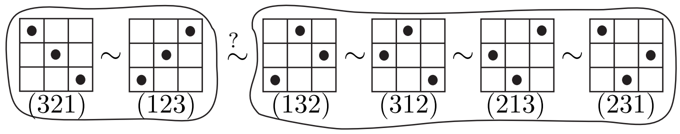

FIGURE 4. Classification of S_3 up to Wilf-equivalence

In Figure 4 we have grouped the six permutations of S_3 into two *symmetry classes*: $\{(321),(123)\}$ and $\{(132),(312),(213),(231)\}$. It is evident that within each such class the permutations are Wilf-equivalent; for instance, if you flip the matrix $M(321)$ across a horizontal axis, you will obtain the matrix $M(123)$; this same flip induces a bijection $S_n(321) \cong S_n(123)$, rendering (321) and (123) as Wilf-equivalent. Similarly, flipping $M(132)$ across a horizontal, vertical or diagonal axis of symmetry produces $M(312)$, $M(213)$ and $M(231)$, and implies $(132) \sim (312) \sim (213) \sim (231)$. In general, acting by the dihedral group (or the symmetries of the square) on the $n \times n$ permutation matrices produces classes of Wilf-equivalent permutations; the orbits of this action are the above-mentioned *symmetry classes* of permutations.

Thus, we have split S_3 into two symmetry classes, and the only thing left to resolve is whether permutations from different classes are *Wilf*-equivalent. At the behest of Wilf, a bijection $S_n(123) \cong S_n(132)$ was suggested by Knuth [14], and shown by Rotem [20], Richards [19], Simion-Schmidt [21], and West [29]. We do not discuss these proofs here, for we shall later, that this situation is a specific instance of a much wider phenomenon. As an exercise, the reader is encouraged to produce one such bijection, or prove that, say, $|S_n(123)| = c_n$. We conclude that

Theorem 1. *S_3 consists of a single Wilf-class, and $|S_n(\tau)| = c_n$ for all $\tau \in S_3$.*

The situation is considerably more complicated already on the level of S_4, not to mention longer permutations. We will return to the classic Wilf-classification in the main part of the paper.

1.4. **Wilf-equivalent pairs.** Nothing prevents us from "forbidding" several permutations at a time. In other words,

Definition 4. *For a collection Ω of permutations (not necessarily of the same length) we denote by $S_n(\Omega)$ the set of permutations in S_n avoiding everything in Ω: $S_n(\Omega) = \cap_{\tau\in\Omega} S_n(\tau)$. If two collections Ω and Υ are equally restrictive on any length permutations, i.e. $|S_n(\Omega)| = |S_n(\Upsilon)|$ for all n, then Ω and Υ are called Wilf-equivalent, which we denote by $\Omega \sim \Upsilon$.*

Considerable attention in the field has been devoted to studying Wilf-equivalent *pairs*. According to a classic result of Erdös and Szekeres [11], the identity permutation $I_k = (1, 2, 3, \ldots, k)$ and its reverse $J_k = (k, k-1, \ldots, 2, 1)$ impose too many conditions on large enough permutations, and hence cannot be avoided simultaneously; more precisely, $|S_n(I_k, J_l)| = 0$ for $n > (k-1)(l-1)$.

More than two decades ago, Simion-Schmidt [21] classified pairs in S_3 up to Wilf-equivalence. Figure 5 displays a representative pair for each of the 5 symmetry classes (pairs are indicated by "+"), and only 2 Wilf-equivalences between these symmetry classes. Thus, for instance, $\{(132), (231)\} \sim \{(132), (213)\} \sim \{(123), (132)\}$ accounts for the first Wilf-class. For each of the resulting 3 Wilf-classes, a number on the side corresponds to $|S_n(\tau_1, \tau_2)|$, e.g. $|S_n((123), (231))| = \binom{n}{2} + 1$. The size 0 for the third Wilf-class is nothing but the above-mentioned Erdös-Szekeres result.

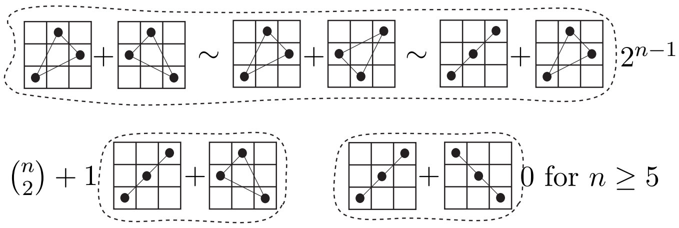

FIGURE 5. Wilf-classes of pairs in S_3

Theorem 2 (Simion-Schmidt). *There are 5 symmetry classes of pairs (τ, σ) of permutations of length 3, with representatives*[3] $(132, 231)$, $(132, 213)$, $(123, 132)$, $(123, 231)$, *and* $(123, 321)$. *There are 3 Wilf-classes of such pairs: the first three symmetry classes join to form one Wilf-class, and the last two symmetry classes stay separate to form two more Wilf-classes. Each Wilf-class produces S_n-subsets of the following sizes:* $|S_n(132, 231)| = 2^{n-1}$ *and* $|S_n(123, 231)| = \binom{n}{2} + 1$ *for all $n \geq 1$, while* $|S_n(123, 321)| = 0$ *for* $n \geq 5$.

The reader is again encouraged to prove these results, whether by direct enumeration of the sizes $|S_n(\tau_1, \tau_2)|$ of each symmetry class, or by finding explicit bijections between these symmetry classes.

A vast amount of research has been generated by the study of Wilf-equivalence of pairs. Not surprisingly, the classification of $(4, 4)$ pairs (i.e. pairs in S_4) already demands a variety of new methods and deeper analysis. What is surprising is that non-trivial Wilf-equivalences among *pairs* seem to occur lot more frequently than among *singleton* of permutations. We shall see examples of this later, but for now it suffices to say that this phenomenon has not been yet explained at all.

[3] For convenience, we have dropped the parentheses around each permutations.

1.5. **Beyond Wilf-equivalence.** Even if we manage to classify all permutations up to Wilf-equivalence, there will still be quite a few questions left to answer. For instance, if $\tau \not\sim \sigma$, then for some n one of the two sets $S_n(\tau)$ and $S_n(\sigma)$ must be smaller, say, $|S_n(\tau)| < |S_n(\sigma)|$. This would mean that τ occurs more often as a subpattern of length-n permutations and hence τ is "harder" to avoid in S_n than σ. Formally,

Definition 5. *If $|S_n(\tau)| \leq |S_n(\sigma)|$ for all $n \in \mathbb{N}$, we say that τ is more restrictive than σ, and denote this by $\tau \preceq \sigma$.*

As usual, let's examine the initial cases of Wilf-ordering. There is nothing to say for S_3, as everything there is Wilf-equivalent to anything else. To talk about Wilf-*ordering* in S_4, we must first understand the Wilf-*equivalences* in S_4.

Theorem 3 (Stankova,West)**.** *There are* 7 *symmetry classes in S_4, whose representatives enter in the following Wilf-equivalences:* $(1234) \sim (1243) \sim (2143) \sim (4123)$*,* $(4132) \sim (3142)$*, and* (1324) *stays separate, for a total of* 3 *Wilf-classes.*

Representatives of each Wilf-class appear in Figure 6a. The Wilf-classification of S_4 was completed over several years by West [29] and Stankova [22, 23]. It required several new methods and is definitely not an easy exercise to offer to the reader for practice. We shall discuss it in detail in a later section. Meanwhile, let's see how the 3 Wilf-classes in S_4 measure against each other. In [5, 7], Bóna provided the only known so far result on complete Wilf-ordering of S_k:

Theorem 4 (Bóna)**.** *The three Wilf-classes in S_4 are ordered as* $(1342) \preceq (1234) \preceq (1324)$*:*

$$|S_n(1342)| < |S_n(1234)| < |S_n(1324)| \text{ for } n \geq 7. \tag{2}$$

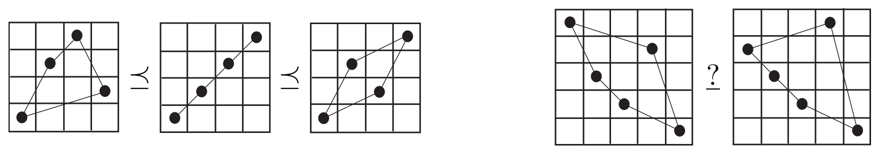

FIGURE 6. Total Wilf-ordering on S_4 and S_5

For each of the two inequalities, Bóna created essentially a new method containing a number of beautiful sophisticated ideas, to which a couple of paragraphs here will not serve justice. More recently, Stankova [25] devised another way of viewing the problem via *decomposable* permutations and generalized Bóna's result to arbitrary lengths.

The point is: even the first non-trivial case of Wilf-ordering generated a great deal of new research, and is indeed far from trivial. The ultimate success in Wilf-ordering S_4 raised hopes for an old conjecture of West, according to which any two permutations (of same length) can be ordered, and hence there is total Wilf-ordering on any S_n. Yet, after years of fruitless search for a proof, a counterexample was found by Stankova already in S_5:

$$S_7(53241) < S_7(43251) \text{ but } S_{13}(53241) > S_{13}(43251),$$

so that these two permutations *cannot* be Wilf-ordered (cf. Fig. 6b). A number of counterexamples were further traced in S_6 and S_7 (cf. [24]), and this completely obviated any hopes for a total Wilf-ordering of a general S_k. What next?

References

[1] M. H. Albert, M. Elder, A. Rechnitzer, P. Westcott, M. Zabrocki, On the Wilf-Stanley limit of 4321-avoiding permutations and a conjecture of Arratia, *Adv. in Appl. Math. 36* (2006), no. 2, 96-105.

[2] R. Arratia, On the Stanley-Wilf Conjecture for the Number of Permutations Avoiding a Given Pattern, *Electron. J. Combin. 6* (1999), no. 1, #1.

[3] E. Babson, J. West, The permutations $123p_4...p_t$ and $321p_4...p_t$ are Wilf-equivalent, *Graphs Comb. 16* (2000) 4, 373-380.

[4] J. Backelin, J. West, G. Xin, Wilf-equivalence for singleton classes, *Proceedings of the 13th Conference on Formal Power Series and Algebraic Combinatorics*, Tempe, AZ, 2001.

[5] M. Bóna, Permutations avoiding certain patterns. The case of length 4 and some generalizations, *Disc. Math. 175* (1997) 55-67.

[6] M. Bóna, The Solution of a Conjecture of Wilf and Stanley for all layered patterns, *J. Combin. Theory, Series A, 85* (1999) 96-104.

[7] M. Bóna, *Combinatorics of Permutations*, Chapman & Hall/CRC, 2004, 135-159.

[8] M. Bóna, The Limit of a Stanley-Wilf sequence is not always rational, and layered patterns beat monotone patterns, *J. Combin. Theory Ser. A 110 (2)* (2005), 223-235.

[9] M. Bóna, "New Records in Stanley-Wilf Limits", submitted to *Electron. J. Combin.*, 2006.

[10] F. Chung, , R. Graham, V. Hoggatt, and M. Kleiman, The number of Baxter permutations, *J. Combin. Theory Ser. A 24, 3* (1978), 382-394.

[11] P. Erdös and G. Szekeres, A combinatorial problem in geometry, *Compos. Math. 2* (1935), 463-470.

[12] I. Gessel, Symmetric functions and P-recursiveness, *J. Combin. Theory Ser. A, 53* (1990), no. 2, 257-285.

[13] V. Jelínek, Dyck paths and pattern-avoiding matchings, *Europ. J. Combin. 28* (2007), no. 1, 202-213.

[14] D. Knuth, *The Art of Computer Programming*, Vol.3, Addison-Wesley, Reading, MA, 1973.

[15] D. Knuth, Permutations, matrices, and generalized Young tableaux, *Pacific J. of Math. 34* (1970) 709-727.

[16] A. Marcus and J. Tardos, Excluded Permutation Matrices and the Stanley-Wilf conjecture, *J. Combin. Theory Ser. A 107* (1) (2004), 153-160.

[17] A. Mier, k-noncrossing and k-nonnesting graphs and fillings of Ferrers diagrams, to appear in Combinatorica.

[18] A. Regev, Asymptotic values for degrees associated with strips of Young diagrams, *Advances in Mathematics 41* (1981), 115-136.

[19] D. Richards, Ballot sequences and restricted permutations, *Ars Combinatoria 25* (1988) 83-86.

[20] D. Rotem, On correspondence between binary trees and a certain type of permutation, *Information Processing Letters 4* (1975), 58-61.

[21] R. Simion and F. Schmidt, Restricted permutations, *Europ. J. Combin. 6* (1985) 383-406.

[22] Z. Stankova, Forbidden subsequences, *Disc. Math. 132* (1994) 291-316.

[23] Z. Stankova, Classification of forbidden subsequences of length 4, *Europ. J. Combin. 17* (1996), 501-517.

[24] Z. Stankova and J. West, A new class of Wilf-equivalent permutations, *J. Algebraic Combin. 15* (2002), no. 3, 271-290.

[25] Z. Stankova, Shape-Wilf-Ordering on Permutations of Length 3, *Electron. J. Combin. 14 (2007), #R56.*

[26] J. Steele, *Probability theory and combinatorial optimization*, *SIAM*, Philadelphia (1997).

[27] J. West, Permutations with forbidden subsequences and stack-sortable permutations, Ph.D. Thesis, M.I.T., Cambridge, MA, 1990.

[28] J. West, Generating trees and the Catalan and Schröder numbers, *Disc. Math. 146* (1995) 247-262.

[29] J. West, Generating trees and forbidden subsequences, *Disc. Math. 157* (1996) 363-374.