Some Problems on Restricted Patterns

by Zvezdelina Stankova

Berkeley Math Circle – Advanced Group

November 26, 2013

(1) How much is $|S_n|$?

(2) List all permutations in $S_n(123)$ and $S_n(231)$ for $n = 1, 2, 3, 4$.

(3) List the symmetry classes of (123) and (132), and verify that they comprise S_3.

(4) Prove that stack-sortable permutations of length n are in 1-1 correspondence with

 (a) Binary strings of n 0's and n 1's where up to any place in the string there are at least as many 0's as 1's.

 (b) The ways to properly parenthesize an expression with n ('s and n)'s.

 (c) Non-diagonal crossing lattice paths from $(0, 0)$ to (n, n).

 (d) 231-avoiding permutations of length n.

(5) For the Catalan numbers, derive the following formulas for all $n \geq 0$:

 (a) $c_{n+1} = c_0c_n + c_1c_{n-1} + c_2c_{n-2} + \cdots + c_{n-2}c_2 + c_{n-1}c_1 + c_nc_0 = \sum_{k=0}^{n} c_kc_{n-k}$, where $c_0 = c_1 = 1$.

 (b) $c_n = \frac{1}{n+1}\binom{2n}{n}$.

(6) Find a 1-1 correspondence between $S_n(123)$ and $S_n(231)$ (or $S_n(132)$).

(7) List the symmetry classes of (1324), (4132), (3142), (2143), (1243), (1234), and (4123), and verify that they comprise S_4.

(8) Draw the generating trees $T(123)$ and $T(132)$ up to level 4 (or 5) and show that they are isomorphic. Do the same for $T(1234)$ and $T(1243)$ up to level 6.

(9) Calculate the first 5 Schröder numbers s_n. Derive the recursive formula:

$$s_n = \sum_{i=1}^{n} s_{i-1}s_{n-i} + s_{n-1},\ s_0 = 1, s_2 = 2.$$

(10) (West, Stankova) Prove that $|S_n(2413, 3142)| = s_{n-1}$ for all $n \geq 1$.

(11) Fix a young diagrams of size 4 and draw all transversals of Y that avoid (123). Do the same thing for (321). Then change Y to another Young diagram of size 4 and repeat. Collect all data and count to show that $|S_Y(123)| = |S_Y(321)|$ for all Y of size 4. Repeat for (213) and (132) to show that $|S_Y(213)| = |S_Y(132)|$ for all Y of size 4.