The Tree of Numbers: Problems

Austin Shapiro • Berkeley Math Circle (Adv) • November 19, 2013

A motivating question

1. I tell you that $\frac{p}{q} = 0.058544\ldots$, where p, q are relatively prime integers and $q < 10^3$. How can you identify p and q?

The tree of numbers

We call a fraction $\frac{p}{q}$ **reduced** if p, q are relatively prime and $q > 0$. Fractions are to be distinguished from rational numbers (but every rational number has a unique reduced form). We condone the fraction $\frac{1}{0}$ and consider it to be reduced.

The **mediant** of the fractions $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{a+c}{b+d}$. Fractions $\frac{a}{b}, \frac{c}{d}$ are **neighborly** if $|ad - bc| = 1$.

2. If $\frac{a}{b} < \frac{c}{d}$, where $a, b, c, d > 0$, show that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.
3. Deduce the **sorting property** of the topograph: all descendants of $\frac{p}{q}$ on the left branch are less than $\frac{p}{q}$ and all descendants on the right branch are greater than $\frac{p}{q}$.
4. If any two of $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{a+c}{b+d}$ are neighborly, show that all three fractions are pairwise neighborly.
5. Show that if two fractions are neighborly, then both are reduced.
6. Deduce that the fractions labeling adjacent faces in the topograph are always neighborly, and that all fractions in the topograph are reduced.
7. Given a reduced fraction $\frac{p}{q}$ with $p, q \neq 0$, show that it has unique "parents" $\frac{a}{b}$ and $\frac{c}{d}$ that are neighborly, reduced, and yield $\frac{p}{q}$ as their mediant. Describe how to find the parents.
8. Show that every reduced fraction appears exactly once in the topograph.

Because of the results above, we can identify the fractions in the topograph with rational numbers (except the mysterious $\frac{1}{0}$). A path from $\frac{1}{1}$ to any positive rational number can be represented as a sequence of L's and R's (left and right descents). For example, the path to $\frac{3}{7}$ is LLRR. A **turn** in the path is where a block of consecutive L's or R's begins.

9. Show that the number of turns in the path from $\frac{1}{1}$ to $\frac{p}{q}$ is at most $2\log_2 q$. How much can this bound be improved?
10. This question is open to interpretation: If we choose our destination $\frac{p}{q}$ at random and write out the path to that destination, what is the "typical" length of a block of consecutive L's or R's?
11. Suppose we follow an infinite path away from $\frac{1}{1}$, and the numbers we encounter are $x_1, x_2, x_3, \ldots$. Show that $\lim_{n\to\infty} x_n$ exists. We define this limit to be the **value** of the path.
12. Show that the value of a path with infinitely many turns is irrational, and that every irrational number is the value of a unique path. Thus, real numbers are in one-to-one correspondence with paths that either terminate or have infinitely many turns.

13. Find some good rational approximations of $\sqrt{2}$ and π ("good" open to interpretation).
14. Characterize the values of paths represented by periodic sequences of L's and R's.

Quadratic forms

A (bivariate) quadratic form is a function $Q : \mathbb{Z}^2 \to \mathbb{Z}$ of the form $Q(x, y) = Ax^2 + Bxy + Cy^2$ (where we will require $A, B, C \in \mathbb{Z}$).

15. Verify these three properties of quadratic forms:
    (i) $Q(-x, -y) = Q(x, y)$
    (ii) $Q(kx, ky) = k^2 Q(x, y)$
    (iii) $Q(x - x', y - y'),\ Q(x, y) + Q(x', y'),\ Q(x + x', y + y')$ are in arithmetic progression

We now rewrite each fraction $\frac{x}{y}$ on the topograph as an ordered pair (x, y), where x and y are relatively prime. We write the value of $Q(x, y)$ on the face labeled by the pair (x, y). Each edge joins two non-adjacent faces; if the values of Q on these two faces are unequal, we orient the edge toward the larger value.

16. Show that if an edge is oriented toward vertex v, then the other two edges incident to v are oriented away from v. This is the **climbing lemma**.
17. Show that if $Q(x, y)$ takes both positive and negative values, but is never 0, then the region of the topograph where Q is positive is separated from the region where Q is negative by an infinite simple path (which Conway calls a **river**). Also, if Q is integer-valued, then the river is periodic when represented as a sequence of L's and R's, and the values of Q at faces along the river are also periodic.

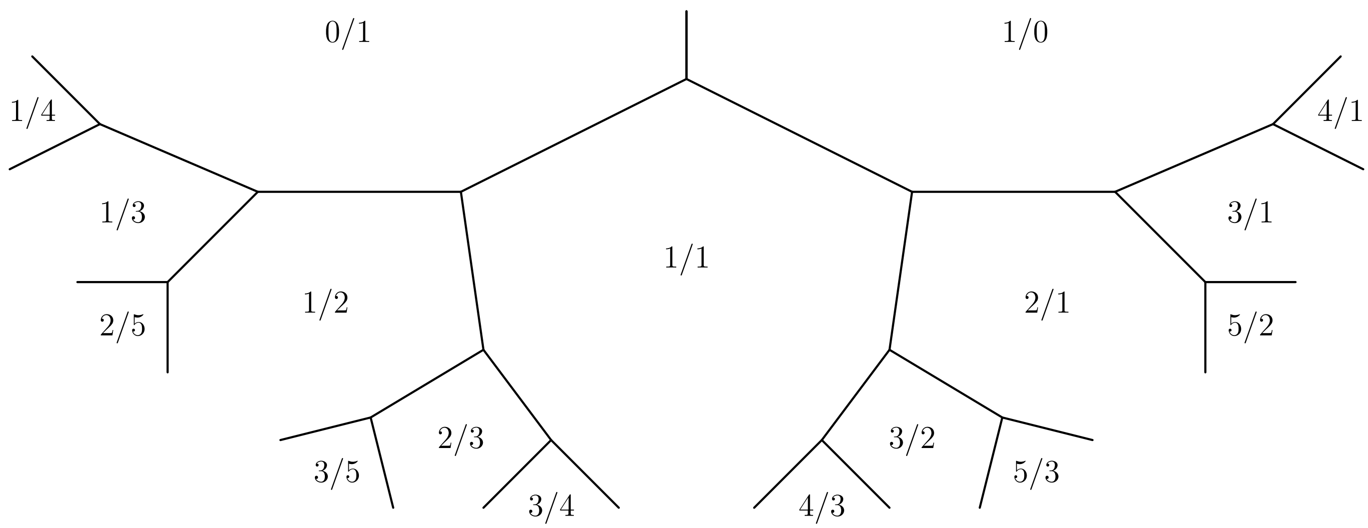
Quadratic forms are classified according to the signs of the values they assume. For example, a form satisfying the hypotheses in the last problem is a $+-$ **form**. The other types of forms are $+$, $+0$, $-$, -0, and $+-0$.

18. If $+-$ forms have a river, what kind of topographic features do the other types of forms have?
19. Devise an algorithm to determine, in finite time, whether the Diophantine equation $Ax^2 + Bxy + Cy^2 = D$ has a solution.
20. Given two quadratic forms Q_1, Q_2, how can we tell whether there is a linear change of coordinates $(x, y) \mapsto (x', y')$ such that $Q_1(x, y) = Q_2(x', y')$ identically?
21. Given a quadratic form Q, how can we determine all linear changes of coordinates $(x, y) \mapsto (x', y')$ for which $Q(x, y) = Q(x', y')$ identically? (Such changes of coordinates are called **isometries**. They form a group.)

Odds and ends

22. For every reduced fraction $\frac{p}{q}$, draw a circle of diameter $\frac{1}{q^2}$ whose lowest point is $\left(\frac{p}{q}, 0\right)$. These are **Ford circles**. Show that the Ford circles corresponding to reduced fractions $\frac{a}{b}$ and $\frac{c}{d}$ are tangent if and only if $\frac{a}{b}, \frac{c}{d}$ are neighborly, and are otherwise nonintersecting.
23. If $\left\{\frac{p_i}{q_i} : 1 \le i \le 2^n\right\}$ are all the n^{th}-generation descendants of $\frac{1}{1}$ in the topograph, show that $\displaystyle\sum_{i=1}^{2^n} \frac{1}{p_i q_i} = 1$.
24. How is the continued fraction form of a rational number related to the continued fraction form of its parent in the topograph?

Further reading: Conway, J. H. *The Sensual (Quadratic) Form.*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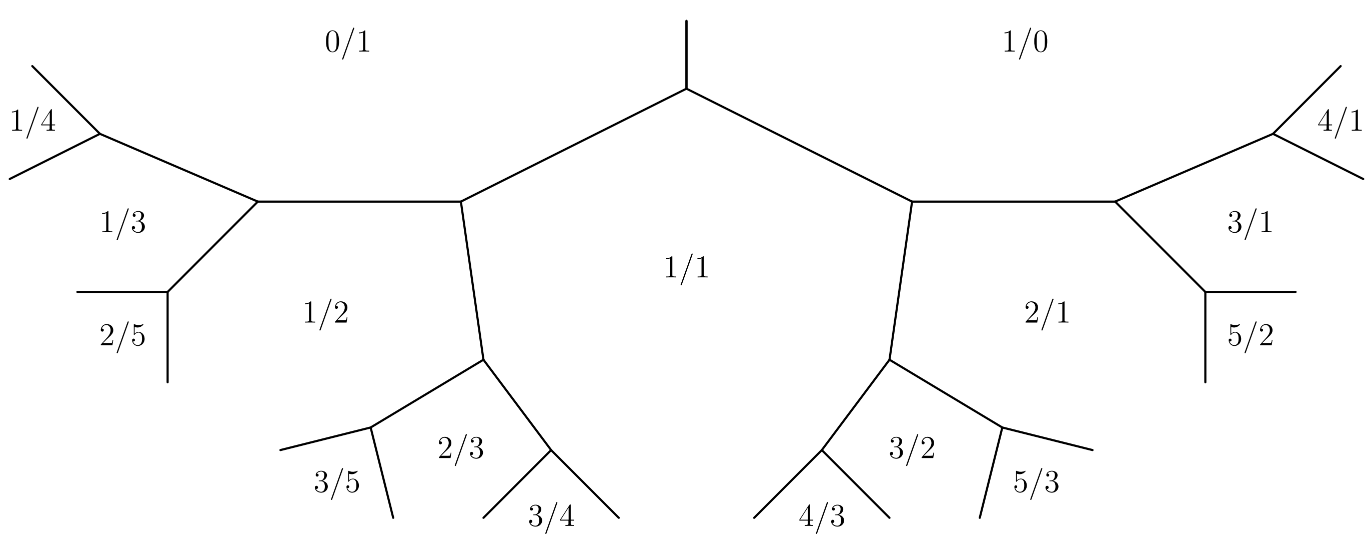
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