BAMO Warmup

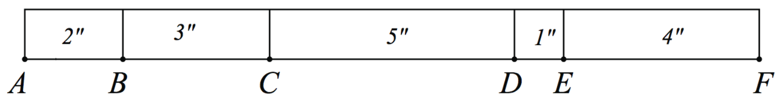
Berkeley Math Circle (Adv) • February 18, 2014

Theme 1

1. *(BAMO 2004, #1b)* A *tiling* of the plane with polygons consists of placing the polygons so that the interiors of polygons do not overlap, each vertex of one polygon coincides with a vertex of another polygon, and no point of the plane is left uncovered. A *unit* polygon is a polygon with all sides of length one.

   Prove that it is impossible to find a tiling of the plane consisting of infinitely many unit squares and finitely many (and at least one) unit equilateral triangles.

2. *(2007, #1)* A 15-inch-long stick has four marks on it, dividing it into five segments of length 1, 2, 3, 4, and 5 inches (although not necessarily in that order) to make a "ruler." Here is an example:



   Using this ruler, you could measure 8 inches (between the marks B and D) and 11 inches (between the end of the ruler at A and the mark at E), but there's no way you could measure 12 inches.

   Prove that it is impossible to place the four marks on the stick such that the five segments have length 1, 2, 3, 4, and 5 inches, and such that every integer distance from 1 inch through 15 inches could be measured.

3. *(2007, #2)* The points of the plane are colored in black and white so that whenever three vertices of a parallelogram are the same color, the fourth vertex is that color, too. Prove that all the points of the plane are the same color.

Theme 2

4. *(2010, #2)* Place eight rooks on a standard 8×8 chessboard so that no two are in the same row or column. With the standard rules of chess this means that no two rooks are attacking each other. Now paint 27 of the remaining squares (not currently occupied by rooks) red.

   Prove that no matter how the rooks are arranged and which set of 27 squares are painted, it is always possible to move some or all of the rooks so that:

   - All the rooks are still on unpainted squares.
   - The rooks are still not attacking each other (no two are in the same row or same column).
   - At least one formerly empty square now has a rook on it; that is, the rooks are not on the same 8 squares as before.

5. *(2005, #3)* Let n be an integer greater than 12. Points $P_1, P_2, \ldots, P_n, Q$ in the plane are distinct. Prove that for some i, at least $n/6 - 1$ of the distances

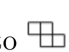
$$P_1P_i, P_2P_i, \ldots, P_{i-1}P_i, P_{i+1}P_i, \ldots, P_nP_i$$

   are less than P_iQ.

6. *(2004, #4)* Suppose one is given n real numbers, not all zero, but such that their sum is zero. Prove that one can label these numbers $a_1, a_2, \ldots, a_n$ in such a manner that

$$a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 < 0.$$

Theme 3

7. *(2008, #2)* Determine the greatest number of figures congruent to  that can be placed in a 9×9 grid (without overlapping), such that each figure covers exactly 4 unit squares.

8. *(2010, #3)* All vertices of a polygon P lie at points with integer coordinates in the plane, and all sides of P have integer lengths. Prove that the perimeter of P must be an even number.

9. *(2006, #5)* We have k switches arranged in a row, and each switch points up, down, left, or right. Whenever three successive switches all point in different directions, all three may be simultaneously turned so as to point in the fourth direction. Prove that this operation cannot be repeated infinitely many times.

Theme 4

10. *(2010, #1)* Suppose a, b, c are real numbers such that $a + b \geq 0$, $b + c \geq 0$, and $c + a \geq 0$. Prove that

$$a + b + c \geq \frac{|a| + |b| + |c|}{3}.$$

11. *(2007, #4)* Let N be the number of ordered pairs (x, y) of integers such that

$$x^2 + xy + y^2 \leq 2007.$$

Remember, integers may be positive, negative, or zero!

(a) Prove that N is odd.

(b) Prove that N is not divisible by 3.

12. *(2010, #5)* Let a, b, c, and d be positive real numbers satisfying $abcd = 1$. Prove that

$$\frac{1}{\sqrt{\frac{1}{2} + a + ab + abc}} + \frac{1}{\sqrt{\frac{1}{2} + b + bc + bcd}} + \frac{1}{\sqrt{\frac{1}{2} + c + cd + cda}} + \frac{1}{\sqrt{\frac{1}{2} + d + da + dab}} \geq \sqrt{2}.$$

Theme 5

13. *(2004, #2)* A given line passes through the center O of a circle. The line intersects the circle at points A and B. Point P lies in the exterior of the circle and does not lie on the line AB. Using only an unmarked straightedge, construct a line through P, perpendicular to the line AB. Give complete instructions for the construction and prove that it works.

14. *(2011, #2)* In a plane, we are given line ℓ, two points A and B neither of which lies on line ℓ, and the reflection A_1 of point A across line ℓ. Using only a straightedge, construct the reflection B_1 of point B across line ℓ. Prove that your construction works.

Note: "Using only a straightedge" means that you can perform only the following operations:

(a) Given two points, you can construct the line through them.

(b) Given two intersecting lines, you can construct their intersection point.

(c) You can select (mark) points in the plane that lie on or off objects already drawn in the plane. (The only facts you can use about these points are which lines they are on or not on.)

15. *(2012, #2)* Laura won the local math olympiad and was awarded a "magical" ruler. With it, she can draw (as usual) lines in the plane, and she can also measure segments and replicate them anywhere in the plane. She can also divide a segment into as many equal parts as she wishes; for instance, she can divide any segment into 17 equal parts.

Laura drew a parallelogram $ABCD$ and decided to try out her magical ruler. With it, she found the midpoint M of side CD, and she extended side CB beyond B to point N so that the segments CB and BN were equal in length. Unfortunately, her mischievous little brother came along and erased everything on Laura's picture except for points A, M, and N. Using Laura's magical ruler, help her reconstruct the original parallelogram $ABCD$: write down the steps that she needs to follow and prove why this will lead to reconstructing the original parallelogram $ABCD$.