Berkeley Math Circle 2014 April 29.

Draft continued fractions notes. (These may not make much sense unless you go to the lecture.)

$\pi = 3.1415926535...$ 3, $22/7 = 3.142857...$ $355/113 = 3.1415929...$ are all good rational approximations.
How do we find good rational approximations to a real number?

Method 1: trial and error works but is slow.

Method 2: repeatedly subtract integer part, take reciprocal.
$3.14159.. \to 7.062 \to 15.9965 \to 1.0034 \to 292.63 \to 1.57...$
giving $\pi = 3 + \cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cfrac{1}{1+...}}}}}$ We can cut off at a large number to get good approximations to π; for example $3 + \cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1}}} = \frac{355}{113}$ gives an unusually good approximation because 292 is large.

Sometimes we get patterns in the continued fraction. For example

$e = 2 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+..}}}}}}}}$ with the numbers going 1 , 2, 1, 1, 4, 1,1, 6, 1, 1, 8, 1, 1, 10, …

This is more dramatic if you take a power of e such as

$e^{1/10} = 1 + \cfrac{1}{9+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{29+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{49+\cfrac{1}{1+..}}}}}}}}$

Exercise: Guess a continued fraction expansion for $e^{1/n}$ and use it to find some good rational approximations to $e^{1/100}$

Exercise: Why does the continued fraction expansion for $e = e^1$ not seem to fit into this pattern?

Exercise. Find the continued fraction for tan(1) (in radians, please!). Can you see a pattern? What about tan(1/10)? Guess a continued fraction for tan(1/x).

Try the simplest possible continued fraction $1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+..}}}}}}}}$ What is its value? If we call it τ we see that $\tau = 1 + 1/\tau$ which is a quadratic equation. Solving it shows that $\tau = \frac{1+\sqrt{5}}{2}$, the famous golden ratio, which has generated more crackpot nonsense than any number other than π. (The pyramids, parthenon and so on were not based on it.)

Notice that the terms of the continued fraction are all 1, which means that τ is the "worst" number of all for rational approximations. Let's find the rational approximations to it: they are 1/1, 2/1, 3/2, 5/3, 8/5, 13/8, 21/13,... Notice that both the numerators and denominators form the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, …. Why is this?

What about $\sqrt{2}$? We have $\sqrt{2} = 1 + \frac{1}{\sqrt{2}+1} = 1 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2+}}}}$

The approximations we get are 1/1, 3/2, 7/5, 17/12, 41/29, … Exercise: What is the pattern of these numbers? Can you prove it?

Another property of these fractions is a little harder to spot: try calculating $a^2 - 2b^2$ for the approximations $\frac{a}{b}$ to $\sqrt{2}$. What pattern do you get?
This suggests a way to solve the Diophantine equation $x^2 - Dy^2 = \pm 1$: try setting $\frac{x}{y}$ equal to an approximation to $\sqrt{D}$ calculated by a continued fraction.

Example: Solve $1 + 61y^2 = x^2$ in positive integers x and y. Solution: $1 + 226153980^2 = 1766319049^2$
We work out the continued fraction:

$$0^2 - 61 \times 1^2 = -61$$
$$1^2 - 61 \times 0^2 = 1$$
$\sqrt{61} = 7 + \frac{1}{\frac{\sqrt{61}+7}{12}}$ $\quad 7^2 - 61 \times 1^2 = -12$
$\frac{\sqrt{61}+7}{12} = 1 + \frac{1}{\frac{\sqrt{61}+5}{3}}$ $\quad 8^2 - 61 \times 1^2 = 3$
$\frac{\sqrt{61}+5}{3} = 4 + \frac{1}{\frac{\sqrt{61}+7}{4}}$ $\quad 39^2 - 61 \times 5^2 = -4$
$\frac{\sqrt{61}+7}{4} = 3 + \frac{1}{\frac{\sqrt{61}+5}{9}}$ $\quad 125^2 - 61 \times 16^2 = 9$
$\frac{\sqrt{61}+5}{9} = 1 + \frac{1}{\frac{\sqrt{61}+4}{5}}$ $\quad 164^2 - 61 \times 21^2 = -5$
$\frac{\sqrt{61}+4}{5} = 2 + \frac{1}{\frac{\sqrt{61}+6}{5}}$ $\quad 453^2 - 61 \times 58^2 = 5$
$\frac{\sqrt{61}+6}{5} = 2 + \frac{1}{\frac{\sqrt{61}+4}{9}}$ $\quad 1070^2 - 61 \times 137^2 = -9$
$\frac{\sqrt{61}+4}{9} = 1 + \frac{1}{\frac{\sqrt{61}+5}{4}}$ $\quad 1523^2 - 61 \times 195^2 = 4$
$\frac{\sqrt{61}+5}{4} = 3 + \frac{1}{\frac{\sqrt{61}+7}{3}}$ $\quad 5639^2 - 61 \times 722^2 = -3$
$\frac{\sqrt{61}+7}{3} = 4 + \frac{1}{\frac{\sqrt{61}+5}{12}}$ $\quad 24079^2 - 61 \times 3083^2 = 12$
$\frac{\sqrt{61}+5}{12} = 1 + \frac{1}{\frac{\sqrt{61}+7}{1}}$ $\quad 29718^2 - 61 \times 3805^2 = -1$
$\sqrt{61} + 7 = 14 + \frac{1}{\frac{\sqrt{61}+7}{12}}$ and repeat.

What is the point of doing such a complicated example? It generates a lot of useful experimental data to examine.
There are a lot of patterns in the data above. For example, notice the "duality" of the coefficients, and the relation between the last number in each line and a denominator (up to sign). There are

recursion formulas for the numerators and denominators: (next thing) equals (coefficient of continued fraction) times (last thing) plus (next to last thing). This holds for all continued fractions and unlike most of the other patterns has nothing to do with square roots.
Exercise: Can you explain why this recursion relation holds? This is harder than most of the other exerecises: one way is to find a formula for the numerators and denominators using 2 by 2 matrices.
The last equation $29718^2 - 61 \times 3805^2 = -1$ is not a solution to our problem because of the wrong sign. We could fix the sign by going through another 11 steps, but there is a short cut. Put $N(x + y\sqrt{61}) = (x + y\sqrt{61})(x - y\sqrt{61}) = x^2 - 61y^2$. Then $N(ab) = N(a)N(b)$ (why?) so if $N(a) = -1$ then $N(a^2 = 1$ So we just calculate $(29718 + \sqrt{61} \times 3805)^2 =$ to solve $x^2 - 61y^2 = 1$.
Actually there is an even easier way, though it is not at all obvious: take the equation $N((39 + 5\sqrt{61})/2) = -1$ and cube it to get $N(29718 + \sqrt{61} \times 3805) = -1$. Why does this work?
Exercise: Show that there are infinitely many solutions to $x^2 - 61y^2 = 1$ in integers.
Exercise. Solve $x^2 - 62y^2 = 1$. This is MUCH easier! 61 is the hardest number less than 100.

Recall that the decimal expansion of a real number is periodic if and only if the number is rational. It is fairly obvious that the continued fraction expansion of a number is finite if and only if the number is rational. A more subtle fact is that the continued fraction expansion is eventually periodic if and only if the number is a root of a quadratic equation with integer coefficients. In one direction this is easy: a periodic continued fraction is a root of a quadratic equation for much the same reason that τ was. In the other direction, look at the calculation for the continued fraction of $\sqrt{61}$. The numbers appearing are all bounded, so must eventually repeat, so the continued fraction is eventually periodic.

What about $\sqrt[3]{2}$? No one has ever found a nice pattern in its continued fraction expansion. Very hard unsolved research problem: Show that the coefficients are unbounded.

How well can we approximate a real number x by a rational number $\frac{a}{b}$ with b at most N? One might guess that since there are about N^2 such numbers (really $3/\pi^2$ times this) between 0 and 1 one should be able to get an error of about $1/N^2$, but this is completely wrong. The point is that the fractions are not at all evenly spaced between 0 and 1. For example, there is a gap of size $1/N$ at 0, and similarly there are large gaps near any rational number of small denominator.

To estimate the error of the best rational approximation to x as above, look at the fractional parts of $0,\ x,\ 2x,\ ...,\ (N)x$. By the pigeonhole principle we can find two of these within distance $1/N$ so $|mx - nx - a| \le 1/N$ for some integers m, n, a. But then $|x - \frac{a}{b}| \le \frac{1}{Nb}$ with $b = m - n$ having absolute value at most N. (Note that the error might be much larger than the naive guess $1/N^2$ if b is small.) In particular we can find infinitely many pairs (a, b) with $|x - \frac{a}{b}| \le \frac{1}{b^2}$ (Notice that this is much better than the naive bound given by fixing b and choosing the best a, which gives a

bound of $1/2b$: cutting off the decimal expansion at some point usually gives a rather poor rational approximation.)
Exercise: check that the good approximations to π satisfy this bound, and find another approximation satisfying it.

The approximations given by continued fractions satisfy this bound. In fact at least one of every two approximations satisfies the stronger bound $|x - \frac{a}{b}| \leq \frac{1}{2b^2}$ and conversely any approximation satisfying this bound is one of the continued fraction approximations.

Further reading: Hardy and Wright, An introduction to the theory of numbers. The chapters on continued fractions and quadratic fields prove some of the facts stated above.